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An Efficient Almon Two-Parameter Estimator for the Heteroscedastic Distributed Lag Model: A Monte Carlo Evidence

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Abstract

The distributed lag models (DLM) are very useful in econometrics and statistics. The technique of Almon polynomial distributed lag is a commonly used estimation method when dealing with the DLM. To address multicollinearity accompanying with the Almon technique, the Almon Two-Parameter Estimator (ATPE) has been recently introduced in the literature, which has some advantages over other available estimators. However, the ATPE can become highly inefficient when the DLM suffers from heteroscedasticity of an unknown form. This study is intended to address this issue and propose an adaptive version of the ATPE which is more effective than the ATPE when the unknown form of heteroscedasticity is present. The performance of the proposed method is evaluated through a Monte Carlo simulation, with mean squared error utilized as an evaluation criterion. These simulation results show the superiority of our suggested method.

Keywords: Almon technique, Almon two-parameter estimator, Distributed lag model, Heteroscedasticity, Multicollinearity.

1. Introduction

Linear regression models are widely used in statistics to gauge the dependence of one variable on one or more independent variables. Such models are used to predict the values of dependent variable by using the known values of independent variable (s). The method of ordinary least squares (OLS) is commonly used estimation method for linear regression models which provide reliable results under certain assumptions about the error term. A distributed lag model (DLM) refers to a linear regression model that incorporates both current and lagged (past) values of an explanatory variable as independent variables. The DLM is frequently used in statistics and econometric to measure the impact of lag values of explanatory variable on the dependent variable. Regarding the regression model, under normal assumptions, the most effective option is the ordinary least square method (OLS) for

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regression parameters estimation. But for DLM, direct application of OLS method is discouraged for two major reasons. Firstly, various lag values of similar explanatory variables are present is this model, thus major issue of multicollinearity might be present and eventually OLS estimator (OLSE) becomes unstable. Secondly, if the size of sample is less than number of lags then there might not be a possibility to estimate parameters with the help of OLS process (Baltagi, 2011; Gujarati, 2003; G. Maddala, 1977). To grab the aforementioned issues, some alternative procedures for the estimation of DLM are proposed in the literature (see, e.g. Almon (1965), Koyck (1954), and Shiller (1973), etc.). Most of these methods rely on prior knowledge of DLM parameters behaviour. This prior knowledge is categorized into two types: stochastic and non-stochastic smoothness priors (Gujarati, 2003; Vinod & Ullah, 1981). The most widely used approach for estimating the DLM was introduced by Almon (1965). The Almon technique assumes that the DLM parameters can be approximated by a suitable-degree polynomial. The assumption simplifies the DLM by transforming it into a model with significantly reducing the number of regressors than the original, making it possible to apply the OLS method to the transformed model.

Although the Almon Estimator (AE) is often a preferred method for estimating the DLM, it can lead to significant multicollinearity among the newly constructed regressors (G. S. Maddala, 1974). To mitigate this issue, the literature recommends combining the Almon technique with biased estimators to address multicollinearity effectively (see, e.g., G. S. Maddala (1974); Chanda and Maddala (1984); Güler et al. (2017); Gültay and Kaçiranlar (2015), and recently Lukman and Kibria (2021)). Additionally, the issue of multicolinearity, the DLM might also encounter the heteroscedasticity problem. In heteroscedasticity's availability, the OLS process added with Almon technique may result inefficient AE and inconsistent covariance matrix of the AE which may cause invalid testing procedures. Recently, this issue attracted the attention of researchers. Majid et al. (2018) addressed the issue of efficient estimation of the DLM while Majid et al. (2017, 2019) incorporated the issue of testing inference in heteroscedasticity's presence of unknown form.

Recently, Özbay and Kaçiranlar (2017) introduced the Almon Two-Parameter Estimator (ATPE) as a solution for estimating the DLM to overcome the issue of multicollinearity. Their results demonstrated that the proposed estimator yields lower mean squared error (MSE) compared to the AE. However, the effectiveness of the ATPE diminishes when the error term of the DLM is heteroscedastic. In cases of heteroscedasticity with an unknown form, the ATPE may become inefficient. This study tackles this problem and proposes an adaptive version of the ATPE which is more effective than the ATPE in heteroscedasticity of unknown form's presence. The superiority of our suggested estimator is justified by a Monte Carlo simulation experiment through comparing MSE.

The remaining article is structured as follows: Section 2 consists of the description of the DLM and the ATPE, provides the adaptive version of the ATPE and estimation methods for the biasing parameters while Section 3 briefly define the Monte Carlo simulation scheme and data generation procedure. Section 4 consists of the results along with discussion while Section 5 concludes the paper.

2. Statistical Methodology

In this section, we present the DLM, the ATPE and our proposed adaptive ATPE which addresses the joint issue of multicollinearity and heteroscedasticity. The estimation methods for the biasing parameters k and d are also discussed.

2.1. The DLM and the ATPE

Finite DLM is given as

$$\begin{aligned} y_t &= \beta_0 x_t + \beta_1 x_{t-1} + \beta_2 x_{t-2} + \cdots + \beta_m x_{t-m} + u_t, \qquad (t = m+1, \cdots, T) \\ &= \sum_{i=0}^{m} \beta_i x_{t-i} + u_t. \end{aligned} \tag{1}$$

The above model demonstrates that the y_t depends not only on the current value of the independent variable but also on specific lagged values of the independent variable. The model expressed in Eq. (1) can be rewritten in matrix form as

$$y = X\beta + u, \tag{2}$$

where

$$y = \begin{bmatrix} y_{m+1} \\ y_{m+2} \\ \vdots \\ y_T \end{bmatrix}, \quad \beta = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{bmatrix}, \quad X = \begin{bmatrix} x_{m+1} & x_m & \ldots & x_1 \\ x_{m+2} & x_{m+1} & \ldots & x_2 \\ \vdots & \vdots & \vdots & \vdots \\ x_T & x_{T-1} & \ldots & x_{T-m} \end{bmatrix}, \quad u = \begin{bmatrix} u_{m+1} \\ u_{m+2} \\ \vdots \\ u_T \end{bmatrix}.$$

It is obvious that y is a $(T-m) \times 1$ vector of observations on independent variable, β is an $(m+1) \times 1$ vector of unknown lag weights or lag coefficients, vector u represents the random error having 0 mean and Ω variance with tth diagonal elements $\sigma_t^2 (t = m+1, m+2, \cdots, T)$ that is, $\Omega = diag\{\sigma_{m+1}^2, \sigma_{m+2}^2, \cdots, \sigma_T^2\}$. Under the condition of homoscedastic error, $\sigma_t^2 \ (t = m+1, \ m+2, \ \cdots, \ T) = \sigma^2$ and $\Omega = \sigma^2 I_{(T-m)}$, where an identity matrix $I_{(T-m)}$ has order $(T-m)$.

The OLSE of parameter vector β in Eq. (2) is $\widehat{\beta}_{OLS} = (X'X)^{-1}Xy$. However, applying the OLS method directly to estimate the model parameters may encounter significant challenges, as previously discussed. A well-known approach to address these challenges is the method of polynomial distributed lag (Almon, 1965) with the assumption that the β_i can be effectively estimated using a polynomial of degree r in i, where $r < s$ (the lag length), that is,

$$\beta_i = \gamma_0 + \gamma_1 i + \gamma_2 i^2 + \cdots + \gamma_k i^k; \qquad i = 0, 1, 2, \cdots, m \text{ and } m \geq k \geq 0. \tag{3}$$

Substituting Eq. (3) in Eq. (1), one can estimate γ by the normal OLS process and then by using Eq. (3), we can gather estimates of β_i. We can write Eq. (3) in matrix notations as

$$\beta = A\gamma, \tag{4}$$

where β is stated above and

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & m & m^2 & \cdots & m^k \end{bmatrix}, \qquad \gamma = \begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \vdots \\ \gamma_k \end{bmatrix}$$

are matrix or order $(m+1) \times (k+1)$ and vector of $(k+1) \times 1$, respectively. By the substitution of Eq. (4) in Eq. (2), we obtain,

$$\begin{aligned} y &= XA\gamma + u, \\ &= Z\gamma + u, \quad \text{such that } Z = XA. \end{aligned} \tag{5}$$

The OLSE of γ is,

$$\widehat{\gamma} = (Z'\,Z)^{-1}\,Z'\,y. \tag{6}$$

The estimate of β is taken as,

$$\widehat{\beta} = A\widehat{\gamma},$$

which is the AE of β. Under usual assumptions of linear regression model, it is the BLUE (Judge et al., 1980) (Judge et al., 1980). A significant strength of the Almon approach is that it assumes the distributed lag follows a polynomial of a specific degree, enabling the use of standard linear regression methods for its estimation (Fair & Jaffee, 1971). Nevertheless, alternative estimation methods have been suggested in the literature to address the multicollinearity issue associated with the Almon technique., (see, for instance, G. S. Maddala (1974); Chanda and Maddala (1984); Güler et al. (2017); Gültay and Kaçiranlar (2015); Özbay and Kaçiranlar (2017)). Adopting the lines of Özkale and Kaçiranlar (2007) in which two-parameter estimator is proposed to cope with multicollinearity issue in linear regression model, Özbay and Kaçiranlar (2017) proposed the ATPE of γ in Model (5), defined as

$$\widehat{\gamma}(h,\ d) = (Z'Z + h\,I)^{-1}\,(Z'\,y\; +\; h\,d\widehat{\gamma}); \quad h > 0,\ 0 < d < 1. \tag{7}$$

Özbay and Kaçiranlar (2017) argue that compared to AE, $\widehat{\gamma}(h,\ d)$ is more stable and shows better performance under few conditions where the Almon technique encounters multicollinearity issues. Thus, the ATPE of parameter vector β is,

$$\widehat{\beta}_{hd} = A\widehat{\gamma}\,(h,\ d)\,. \tag{8}$$

2.2. Proposed adaptive ATPE

The ATPE may be severely inefficient when error term in the DLM exhibits heteroscedasticity of unknown form. Thus an effective ATPE of γ is required that further produces efficient estimator of β when the issues of multicollinearity and heteroscedasticity are coupled together. With reference to the study of Carroll

(1982) and Carroll and Ruppert (1982), we propose an adaptive version of the ATPE to get efficient estimates of lag coefficients in presence of both multicollinearity and heteroscedasticity. A colossal literature is available to justify the similar adaptation for the heteroscedastic linear regression model, see, e.g. Majid et al. (2018), Aslam (2014), Aslam et al. (2013), Ahmed et al. (2011), Carroll (1982), etc. We can define a weighted version of the ATPE for parameter vector γ as

$$\widehat{\gamma}_W(h,\, d) = (Z'\,W\,Z + h\,I)^{-1}(Z'\,W\,y + h\,d\,\widehat{\gamma}_{WLS}); \quad h > 0,\; 0 < d < 1. \tag{9}$$

where $\widehat{\gamma}_{WLS}$ is the weighted least square (WLS) estimator of γ with $W = diag\{w_{m+1}, w_{m+2}, \ldots, w_T\}$ and $w_t = \sigma_t^{-2}$.

Now, main concern is the estimation of weights $w_t = \sigma_t^{-2}$. Based on Carroll's 1982 work, we propose an adaptive estimator to estimate W. Following Carroll's 1982 work, we assume that the error variance is a smooth function of the mean values, expressed as

$$\sigma_t^2 = f\left(\tau_t\right),$$

where f is unknown and estimate of $\tau_t = Z_t'\gamma$, in respect of Eq. (5), is defined as

$$g_t = Z_t'\widehat{\gamma}\left(h,\, d\right).$$

Carroll (1982) proposed the kernel estimator of σ_t^2 using the form of estimator proposed by Nadaraya (1964) as:

$$\widehat{\sigma}_t^2 = \frac{\sum_{j=m+1}^{T} K\left(\frac{g_j - g_t}{C}\right)\hat{u}_j^2}{\sum_{j=m+1}^{T} K\left(\frac{g_j - g_t}{C}\right)},$$

where kernel function is $K(\cdot)$ with smoothing parameter C, and $\hat{u}_j$ denotes the residuals from the ATPE model in Eq. (5).

Now, the adaptive ATPE (AATPE) for the parameter vector γ is given as

$$\widehat{\gamma}_{\hat{W}}(h,\, d) = (Z'\,\hat{W}\,Z\, +\, hI)^{-1}\,(Z'\,\hat{W}\,y + h\,d\,\widehat{\gamma}_{AWLS}); \quad h > 0,\; 0 < d < 1. \tag{10}$$

where $\widehat{\gamma}_{AWLS}$ is the adaptive weighted least square (AWLS) estimator of γ as proposed by Majid et al. (2018) and $\hat{W} = diag\{\hat{w}_{m+1}, \hat{w}_{m+2}, \ldots, \hat{w}_T\}$ with $\hat{w}_t = \widehat{\sigma}_t^{-2}$. Thus, the AATPE of parameter vector β can be defined as

$$\widetilde{\beta}_{hd} = A\widehat{\gamma}_{\hat{W}}\left(h,\, d\right).$$

The estimator $\widetilde{\beta}_{hd}$ would be more efficient than $\widehat{\beta}_{hd}$ when the joint issue of multicollinearity and heteroscedasticity is associated with the Almon technique.

2.3. Estimation of Biasing Parameters h and d

One main concern while using ATPE is the selection of h and d. To find a proper estimate for h and d, Özbay and Kaçiranlar (2017) proposed an iterative method. Following estimates for the ridge parameter were proposed by them for a fixed value of d

$$\hat{h}_{HM} = \frac{(k+1)\hat{\sigma}_u^2}{\sum_{i=1}^{k+1}[\hat{\gamma}_i^2 - d(\hat{\sigma}_u^2/\lambda_i + \hat{\gamma}_i^2)]},$$
$$\hat{h}_{AM} = \frac{1}{(k+1)}\sum_{i=1}^{k+1}\frac{\hat{\sigma}_u^2}{[\hat{\gamma}_i^2 - d(\hat{\sigma}_u^2/\lambda_i + \hat{\gamma}_i^2)]},$$
$$\hat{h}_{GM} = \frac{\hat{\sigma}_u^2}{\prod_{i=1}^{k+1}[\hat{\gamma}_i^2 - d(\hat{\sigma}_u^2/\lambda_i + \hat{\gamma}_i^2)]^{\frac{1}{(k+1)}}},$$

where $\hat{\gamma}$ and $\hat{\sigma}_u^2$ are the γ and OLS estimates of σ_t^2, respectively. For a fixed h value, they also suggested the optimal estimator of d as,

$$\hat{d}_{opt} = \frac{\sum_{i=1}^{k+1}(h\hat{\gamma}_i^2 - \hat{\sigma}_u^2)/(\lambda_i + h)^2}{\sum_{i=1}^{k+1}h(\hat{\sigma}_u^2 + \hat{\gamma}_i^2\lambda_i)/\lambda_i(\lambda_i + h)^2}. \tag{11}$$

Consequently, h and d estimates can be derived using the following iterative method. At first, the estimates of $\hat{h}_{AM}$, $\hat{h}_{HM}$, and $\hat{h}_{GM}$ are obtained by using appropriate estimate of d; $\hat{d} < \min\{\frac{\hat{\gamma}_i^2}{(\hat{\sigma}_u^2/\lambda_i + \hat{\gamma}_i^2)}\}$. The $\hat{d}_{opt}$ is obtained from Eq. (11) by using $\hat{h}_{AM}$, $\hat{h}_{HM}$, and $\hat{h}_{GM}$ estimators. Finally, if $\hat{d}_{opt}$ is negative, $\hat{d} = \hat{d}_{opt}$ is used.

Similarly, for the case of AATPE, we replace $\hat{\sigma}_u^2$ by

$$\hat{\sigma}_{AWLS}^2 = \frac{\sum_{t=m+1}^{T}\hat{u}_{AWLS_t}^2}{(T-m)-(m+1)}$$

where $\hat{u}_{AWLS_t}^2$ are the AWLS residuals and $\hat{\gamma}_i^2$ by $\hat{\gamma}_{AWLS_i}^2$. For further detail about $\hat{\sigma}_{AWLS}^2$ and $\hat{\gamma}_{AWLS_i}^2$, one can see Majid et al. (2018).

3. Empirical study

For numerical evaluation, we utilized the simulation framework developed by Frost (1975), as implemented in the studies by Özbay and Kaçiranlar (2017), Majid et al. (2018, 2019) and Majid and Aslam (2023). The observations are generated by the following model

$$y_t = \beta_0 x_t + \beta_1 x_{t-1} + \beta_2 x_{t-2} + \cdots + \beta_m x_{t-m} + u_t, \tag{12}$$
$$x_1 = v_1, \tag{13.a}$$
$$x_t = x_{t-1} + (1-\lambda^2)^{1/2} v_t \quad \text{for } t \geq 2, \tag{13.b}$$

where $u_t = \sigma_t \epsilon_t$ and ϵ_t are independent standard normal having standard deviation 1 and mean 0. Alternatively, $u_t = \sigma_t\epsilon_t$ is distributed having 0 mean and σ_t standard deviation. In order to see the behaviour of estimators for non-normal errors, we use two more error terms which are distributed as $t(6)$ and $F(4, 16)$. Following Özbay and Kaçiranlar (2017), the lag length $m = 10$ and polynomial degree $k = 2$ is used. The coefficient vector β is selected as the normalized eigenvector associated with the largest eigenvalue of $X'X$ so that $\beta'\beta = 1$ (see Kibria, 2003). The variable v_t follows an independent standard normal distribution with a mean 0 and a standard deviation 1. The coefficient λ represents the expected correlation between consecutive values of x_t. We use four separate values of λ as 0.8, 0.9, 0.95, and 0.99. As λ increases, the degree of multicollinearity is expected to rise, prompting an investigation into how estimators behave under higher levels of multicollinearity. For each replication, this data is obtained by drawing the v_t values for $t = 1, 2, 3, \cdots, 40$ and u_t for $t = 11, 12, 13, \cdots, 40$. The x_t values are obtained by Eq. (13.a) and (13.b) at the beginning of simulation and kept fixed. The y_t values are generated by equation (12) and replicated through simulations. Each replication involves running a regression for $T = 30$ observations (from $t = 11$ to $t = 40$), allowing for lags of up to 10 periods. The generate x_t observations for $T = 30$ are then repeated twice and four times to create larger samples with $T = 60$ and $T = 120$ observations, respectively. These larger samples are constructed to ensure that the degree of heteroscedasticity remains consistent across all sample sizes (see Aslam, 2014; Cribari-Neto, 2004; Majid et al., 2019, in a way for getting larger samples). After getting matrix X, the Z is derived by $Z = X\,A$. After constructing the X and Z matrices, they are held constant throughout the simulations.

Following the work of Aslam (2014), Cribari-Neto and da Silva (2011), and Majid et al. (2019), the error term variance is obtained as:

$$\sigma_t^2 = \mathrm{e}^{\theta_0 x_t + \theta_1 x_{t-1} + \cdots + \theta_{10} x_{t-10}}, \qquad \text{where } t = 11,\ 12,\ 13,\ \cdots,\ 49,\ 50.$$

Here, $\theta = \theta_0 = \theta_1 = \cdots = \theta_{10}$. Heteroscedasticity degree is calculated by $\delta = \frac{\max(\sigma_t^2)}{\min(\sigma_t^2)}$. Heteroscedasticity degree serves as a useful indicator for measuring the intensity of heteroscedasticity, ranging from mild to severe. This term has been extensively used in the research of several authors, including Ahmed et al. (2011), Aslam (2014), Cribari-Neto and da Silva (2011), and Cribari-Neto and Lima (2009), among others. For each value of λ, the θ value is selected such that mild, moderate, and severe heteroscedasticity corresponds to $\delta \approx 4, 36$ and 100, respectively. Clearly, when $\delta = 1$, $\theta = 0$ indicates homoscedasticity in the error term.

The total number of repetitions for the Monte Carlo experiment is set to 5000. All calculations are carried out using R (R Core Team, 2024). Following the approach of Aslam (2014) and Majid et al. (2018), the *npreg*{} procedure in R is used for kernel estimation of the error variance. This method employs kernel and optimal bandwidth selection based on the procedures outlined by Li and Racine (2004) and Racine and Li (2004).

To evaluate the efficiency of both the conventional ATE and the proposed AATE in the presence of both multicollinearity and heteroscedasticity, the MSE criterion is used, focusing on the estimator that yields a lower MSE. For any given estimator $\widehat{\beta}$ of β, the MSE is defined as

$$\text{MSE}\left(\widehat{\beta}\right) = \text{E}\left(\widehat{\beta} - \beta\right)'\left(\widehat{\beta} - \beta\right).$$

But, for the purpose of evaluation, the estimated MSE can be stated as

$$\text{EMSE}(\widehat{\beta}) = \frac{\sum\limits_{n=1}^{N} (\widehat{\beta}_{(n)} - \beta)'(\widehat{\beta}_{(n)} - \beta)}{N},$$

where N denotes the total number of simulation repetitions and $\widehat{\beta}_{(n)}$ represents the estimate of β in n^{th} repetition. The results obtained from simulation are listed in Tables 1–3.

Table 1: Estimated MSE of the ATPE and AATPE when error term follows $N(0,1)$.

δ	λ	$\hat{h}, \hat{d}$	$T = 30$		$T = 60$		$T = 120$	
			ATPE	AATPE	ATPE	AATPE	ATPE	AATPE
1	0.80	$\hat{h}_{HM}, \hat{d}$	0.0095	0.0101	0.0044	0.0045	0.0044	0.0042
		$\hat{h}_{AM}, \hat{d}$	0.0119	0.0127	0.0058	0.0061	0.0151	0.0169
		$\hat{h}_{GM}, \hat{d}$	0.0094	0.0099	0.0040	0.0042	0.0055	0.0059
	0.90	$\hat{h}_{HM}, \hat{d}$	0.0150	0.0156	0.0052	0.0053	0.0025	0.0025
		$\hat{h}_{AM}, \hat{d}$	0.0195	0.0205	0.0098	0.0102	0.0059	0.0061
		$\hat{h}_{GM}, \hat{d}$	0.0157	0.0162	0.0059	0.0060	0.0028	0.0029
	0.95	$\hat{h}_{HM}, \hat{d}$	0.0277	0.0288	0.0094	0.0096	0.0043	0.0044
		$\hat{h}_{AM}, \hat{d}$	0.0332	0.0344	0.0163	0.0168	0.0107	0.0110
		$\hat{h}_{GM}, \hat{d}$	0.0289	0.0297	0.0110	0.0113	0.0057	0.0058
	0.99	$\hat{h}_{HM}, \hat{d}$	0.1537	0.1631	0.0743	0.0778	0.0346	0.0353
		$\hat{h}_{AM}, \hat{d}$	0.1495	0.1553	0.0765	0.0791	0.0526	0.0531
		$\hat{h}_{GM}, \hat{d}$	0.1471	0.1531	0.0733	0.0759	0.0376	0.0381
4	0.80	$\hat{h}_{HM}, \hat{d}$	0.0176	0.0152	0.0064	0.0057	0.0044	0.0042
		$\hat{h}_{AM}, \hat{d}$	0.0196	0.0186	0.0083	0.0082	0.0151	0.0169
		$\hat{h}_{GM}, \hat{d}$	0.0172	0.0158	0.0061	0.0059	0.0055	0.0059
	0.90	$\hat{h}_{HM}, \hat{d}$	0.0326	0.0292	0.0089	0.0082	0.0038	0.0035
		$\hat{h}_{AM}, \hat{d}$	0.0368	0.0357	0.0140	0.0143	0.0085	0.0089
		$\hat{h}_{GM}, \hat{d}$	0.0332	0.0319	0.0099	0.0101	0.0045	0.0050
	0.95	$\hat{h}_{HM}, \hat{d}$	0.0871	0.0738	0.0239	0.0223	0.0106	0.0106
		$\hat{h}_{AM}, \hat{d}$	0.0914	0.0882	0.0302	0.0305	0.0182	0.0187
		$\hat{h}_{GM}, \hat{d}$	0.0880	0.0854	0.0257	0.0270	0.0128	0.0147
	0.99	$\hat{h}_{HM}, \hat{d}$	2.3603	1.5632	0.9187	0.5671	0.4287	0.2713
		$\hat{h}_{AM}, \hat{d}$	2.0968	2.1259	0.8082	0.8006	0.3794	0.3761
		$\hat{h}_{GM}, \hat{d}$	2.1229	2.1263	0.8174	0.7998	0.3823	0.3751
36	0.80	$\hat{h}_{HM}, \hat{d}$	0.0655	0.0309	0.0157	0.0074	0.0063	0.0034
		$\hat{h}_{AM}, \hat{d}$	0.0639	0.0400	0.0169	0.0122	0.0076	0.0066
		$\hat{h}_{GM}, \hat{d}$	0.0624	0.0377	0.0148	0.0097	0.0058	0.0045
	0.90	$\hat{h}_{HM}, \hat{d}$	0.1479	0.0858	0.0256	0.0140	0.0111	0.0069
		$\hat{h}_{AM}, \hat{d}$	0.1358	0.1165	0.0297	0.0215	0.0164	0.0142
		$\hat{h}_{GM}, \hat{d}$	0.1328	0.1026	0.0264	0.0188	0.0123	0.0109
	0.95	$\hat{h}_{HM}, \hat{d}$	0.7616	0.4024	0.1490	0.0674	0.0546	0.0279

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δ	λ	$\hat{h}, \hat{d}$	$T=30$		$T=60$		$T=120$	
			ATPE	AATPE	ATPE	AATPE	ATPE	AATPE
		$\hat{h}_{AM}, \hat{d}$	0.7549	0.6396	0.1456	0.0985	0.0580	0.0386
		$\hat{h}_{GM}, \hat{d}$	0.7554	0.6385	0.1449	0.0978	0.0552	0.0378
	0.99	$\hat{h}_{HM}, \hat{d}$	221.2467	104.3534	69.4568	27.6793	31.6997	11.0292
		$\hat{h}_{AM}, \hat{d}$	196.0180	170.4857	58.1893	47.3568	26.2248	19.2366
		$\hat{h}_{GM}, \hat{d}$	199.4210	170.3301	59.5776	47.2707	26.8596	19.1912
100	0.80	$\hat{h}_{HM}, \hat{d}$	0.1364	0.0588	0.0253	0.0100	0.0101	0.0043
		$\hat{h}_{AM}, \hat{d}$	0.1305	0.0816	0.0245	0.0154	0.0109	0.0084
		$\hat{h}_{GM}, \hat{d}$	0.1294	0.0798	0.0230	0.0133	0.0092	0.0063
	0.90	$\hat{h}_{HM}, \hat{d}$	0.3350	0.1540	0.0490	0.0205	0.0195	0.0102
		$\hat{h}_{AM}, \hat{d}$	0.3346	0.2285	0.0512	0.0307	0.0239	0.0173
		$\hat{h}_{GM}, \hat{d}$	0.3330	0.2269	0.0490	0.0289	0.0206	0.0148
	0.95	$\hat{h}_{HM}, \hat{d}$	2.4017	1.0774	0.4056	0.1288	0.1452	0.0437
		$\hat{h}_{AM}, \hat{d}$	2.3693	1.8288	0.3809	0.2124	0.1367	0.0627
		$\hat{h}_{GM}, \hat{d}$	2.3835	1.8270	0.3872	0.2118	0.1377	0.0623
	0.99	$\hat{h}_{HM}, \hat{d}$	2000.2610	802.1102	561.5720	179.1511	254.9530	63.1918
		$\hat{h}_{AM}, \hat{d}$	1759.3900	1319.1450	466.4076	314.3280	208.5307	113.5723
		$\hat{h}_{GM}, \hat{d}$	1793.0670	1317.7600	479.9732	313.7065	214.7479	113.3003

Table 2: Estimated MSE of the ATPE and AATPE when error term follows $t(6)$.

δ	λ	$\hat{h}, \hat{d}$	$T=30$		$T=60$		$T=120$	
			ATPE	AATPE	ATPE	AATPE	ATPE	AATPE
1	0.8	$\hat{h}_{HM}, \hat{d}$	0.0096	0.0097	0.0043	0.0043	0.0026	0.0026
		$\hat{h}_{AM}, \hat{d}$	0.0119	0.0123	0.0058	0.0060	0.0028	0.0029
		$\hat{h}_{GM}, \hat{d}$	0.0094	0.0094	0.0039	0.0039	0.0021	0.0020
	0.9	$\hat{h}_{HM}, \hat{d}$	0.0154	0.0156	0.0053	0.0053	0.0024	0.0024
		$\hat{h}_{AM}, \hat{d}$	0.0200	0.0205	0.0098	0.0101	0.0059	0.0061
		$\hat{h}_{GM}, \hat{d}$	0.0162	0.0162	0.0059	0.0059	0.0028	0.0028
	0.95	$\hat{h}_{HM}, \hat{d}$	0.0278	0.0282	0.0096	0.0095	0.0042	0.0042
		$\hat{h}_{AM}, \hat{d}$	0.0336	0.0340	0.0165	0.0167	0.0108	0.0110
		$\hat{h}_{GM}, \hat{d}$	0.0291	0.0291	0.0113	0.0111	0.0057	0.0057
	0.99	$\hat{h}_{HM}, \hat{d}$	0.1614	0.1657	0.0748	0.0751	0.0359	0.0358
		$\hat{h}_{AM}, \hat{d}$	0.1576	0.1565	0.0773	0.0768	0.0420	0.0418
		$\hat{h}_{GM}, \hat{d}$	0.1552	0.1542	0.0742	0.0736	0.0379	0.0376
4	0.8	$\hat{h}_{HM}, \hat{d}$	0.0182	0.0157	0.0063	0.0054	0.0033	0.0030
		$\hat{h}_{AM}, \hat{d}$	0.0202	0.0191	0.0080	0.0079	0.0039	0.0039
		$\hat{h}_{GM}, \hat{d}$	0.0178	0.0163	0.0059	0.0056	0.0028	0.0026
	0.9	$\hat{h}_{HM}, \hat{d}$	0.0331	0.0284	0.0080	0.0072	0.0038	0.0034
		$\hat{h}_{AM}, \hat{d}$	0.0372	0.0349	0.0132	0.0135	0.0084	0.0089
		$\hat{h}_{GM}, \hat{d}$	0.0337	0.0310	0.0090	0.0092	0.0045	0.0050
	0.95	$\hat{h}_{HM}, \hat{d}$	0.0849	0.0714	0.0226	0.0204	0.0103	0.0102
		$\hat{h}_{AM}, \hat{d}$	0.0892	0.0854	0.0292	0.0287	0.0178	0.0184
		$\hat{h}_{GM}, \hat{d}$	0.0859	0.0826	0.0246	0.0251	0.0125	0.0143
	0.99	$\hat{h}_{HM}, \hat{d}$	2.3321	1.4820	0.9346	0.5676	0.4200	0.2573
		$\hat{h}_{AM}, \hat{d}$	2.0764	2.0215	0.8288	0.7974	0.3718	0.3573

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δ	λ	$\hat{h}, \hat{d}$	$T = 30$		$T = 60$		$T = 120$	
			ATPE	AATPE	ATPE	AATPE	ATPE	AATPE
		$\hat{h}_{GM}, \hat{d}$	2.1029	2.0222	0.8366	0.7964	0.3740	0.3563
	0.8	$\hat{h}_{HM}, \hat{d}$	0.0691	0.0313	0.0152	0.0070	0.0064	0.0033
		$\hat{h}_{AM}, \hat{d}$	0.0681	0.0405	0.0164	0.0118	0.0076	0.0065
		$\hat{h}_{GM}, \hat{d}$	0.0664	0.0382	0.0143	0.0093	0.0059	0.0045
	0.9	$\hat{h}_{HM}, \hat{d}$	0.1481	0.0781	0.0262	0.0146	0.0105	0.0067
		$\hat{h}_{AM}, \hat{d}$	0.1506	0.1045	0.0307	0.0224	0.0159	0.0140
36		$\hat{h}_{GM}, \hat{d}$	0.1482	0.1024	0.0273	0.0198	0.0118	0.0108
	0.95	$\hat{h}_{HM}, \hat{d}$	0.7864	0.3935	0.1409	0.0629	0.0553	0.0288
		$\hat{h}_{AM}, \hat{d}$	0.7803	0.6327	0.1401	0.0933	0.0593	0.0399
		$\hat{h}_{GM}, \hat{d}$	0.7807	0.6316	0.1390	0.0926	0.0564	0.0392
	0.99	$\hat{h}_{HM}, \hat{d}$	212.3086	91.1246	73.8004	28.5977	30.1717	10.2192
		$\hat{h}_{AM}, \hat{d}$	186.5094	151.1362	63.0564	50.1344	24.9597	17.6925
		$\hat{h}_{GM}, \hat{d}$	189.6522	150.9757	64.3231	50.0474	25.5523	17.6493
	0.8	$\hat{h}_{HM}, \hat{d}$	0.1405	0.0579	0.0266	0.0100	0.0099	0.0042
		$\hat{h}_{AM}, \hat{d}$	0.1352	0.0801	0.0261	0.0157	0.0109	0.0083
		$\hat{h}_{GM}, \hat{d}$	0.1340	0.0783	0.0245	0.0136	0.0091	0.0063
	0.9	$\hat{h}_{HM}, \hat{d}$	0.3294	0.1530	0.0495	0.0219	0.0187	0.0102
		$\hat{h}_{AM}, \hat{d}$	0.3324	0.2314	0.0529	0.0325	0.0236	0.0172
100		$\hat{h}_{GM}, \hat{d}$	0.3303	0.2299	0.0504	0.0309	0.0201	0.0149
	0.95	$\hat{h}_{HM}, \hat{d}$	2.5781	1.1375	0.4022	0.1277	0.1469	0.0446
		$\hat{h}_{AM}, \hat{d}$	2.5590	1.9508	0.3837	0.2091	0.1414	0.0664
		$\hat{h}_{GM}, \hat{d}$	2.5702	1.9491	0.3883	0.2085	0.1418	0.0661
	0.99	$\hat{h}_{HM}, \hat{d}$	1945.8080	773.6727	586.6317	181.4460	232.6627	60.3554
		$\hat{h}_{AM}, \hat{d}$	1705.7090	1301.7960	487.8811	329.3681	190.4486	107.5861
		$\hat{h}_{GM}, \hat{d}$	1736.3310	1300.5560	500.1131	328.8072	195.6488	107.3382

Table 3: Estimated MSE of the ATPE and AATPE when error term follows $F(4, 16)$.

δ	λ	$\hat{h}, \hat{d}$	$T = 30$		$T = 60$		$T = 120$	
			ATPE	AATPE	ATPE	AATPE	ATPE	AATPE
	0.8	$\hat{h}_{HM}, \hat{d}$	0.0091	0.0087	0.0044	0.0043	0.0026	0.0026
		$\hat{h}_{AM}, \hat{d}$	0.0116	0.0115	0.0060	0.0060	0.0028	0.0028
		$\hat{h}_{GM}, \hat{d}$	0.0090	0.0084	0.0041	0.0039	0.0020	0.0020
	0.9	$\hat{h}_{HM}, \hat{d}$	0.0151	0.0143	0.0052	0.0048	0.0025	0.0024
		$\hat{h}_{AM}, \hat{d}$	0.0196	0.0193	0.0098	0.0098	0.0059	0.0060
1		$\hat{h}_{GM}, \hat{d}$	0.0158	0.0149	0.0059	0.0054	0.0028	0.0027
	0.95	$\hat{h}_{HM}, \hat{d}$	0.0269	0.0255	0.0101	0.0093	0.0043	0.0041
		$\hat{h}_{AM}, \hat{d}$	0.0329	0.0315	0.0169	0.0165	0.0109	0.0110
		$\hat{h}_{GM}, \hat{d}$	0.0284	0.0265	0.0117	0.0109	0.0058	0.0056
	0.99	$\hat{h}_{HM}, \hat{d}$	0.1600	0.1520	0.0694	0.0641	0.0372	0.0351
		$\hat{h}_{AM}, \hat{d}$	0.1563	0.1426	0.0727	0.0666	0.0434	0.0413
		$\hat{h}_{GM}, \hat{d}$	0.1538	0.1403	0.0694	0.0632	0.0393	0.0370
	0.8	$\hat{h}_{HM}, \hat{d}$	0.0178	0.0158	0.0059	0.0053	0.0034	0.0031
		$\hat{h}_{AM}, \hat{d}$	0.0198	0.0193	0.0078	0.0078	0.0040	0.0039

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δ	λ	$\hat{h}, \hat{d}$	$T = 30$ ATPE	$T = 30$ AATPE	$T = 60$ ATPE	$T = 60$ AATPE	$T = 120$ ATPE	$T = 120$ AATPE
		$\hat{h}_{GM}, \hat{d}$	0.0174	0.0165	0.0057	0.0055	0.0028	0.0027
	0.9	$\hat{h}_{HM}, \hat{d}$	0.0332	0.0296	0.0084	0.0075	0.0038	0.0033
		$\hat{h}_{AM}, \hat{d}$	0.0370	0.0361	0.0134	0.0136	0.0085	0.0088
		$\hat{h}_{GM}, \hat{d}$	0.0336	0.0324	0.0093	0.0094	0.0045	0.0049
	0.95	$\hat{h}_{HM}, \hat{d}$	0.0878	0.0747	0.0234	0.0205	0.0106	0.0104
		$\hat{h}_{AM}, \hat{d}$	0.0919	0.0888	0.0296	0.0284	0.0180	0.0185
		$\hat{h}_{GM}, \hat{d}$	0.0887	0.0861	0.0252	0.0250	0.0128	0.0145
	0.99	$\hat{h}_{HM}, \hat{d}$	2.2960	1.3080	0.9289	0.5191	0.4267	0.2600
		$\hat{h}_{AM}, \hat{d}$	2.0384	1.7798	0.8202	0.7252	0.3809	0.3552
		$\hat{h}_{GM}, \hat{d}$	2.0623	1.7812	0.8284	0.7244	0.3833	0.3543
	0.8	$\hat{h}_{HM}, \hat{d}$	0.0689	0.0432	0.0154	0.0096	0.0062	0.0039
		$\hat{h}_{AM}, \hat{d}$	0.0676	0.0543	0.0164	0.0139	0.0076	0.0067
		$\hat{h}_{GM}, \hat{d}$	0.0661	0.0523	0.0146	0.0118	0.0058	0.0049
	0.9	$\hat{h}_{HM}, \hat{d}$	0.1480	0.0941	0.0251	0.0169	0.0104	0.0077
		$\hat{h}_{AM}, \hat{d}$	0.1502	0.1265	0.0291	0.0247	0.0158	0.0147
36		$\hat{h}_{GM}, \hat{d}$	0.1478	0.1245	0.0259	0.0224	0.0117	0.0117
	0.95	$\hat{h}_{HM}, \hat{d}$	0.7893	0.4076	0.1434	0.0723	0.0545	0.0304
		$\hat{h}_{AM}, \hat{d}$	0.7843	0.6473	0.1431	0.1073	0.0589	0.0424
		$\hat{h}_{GM}, \hat{d}$	0.7850	0.6461	0.1417	0.1066	0.0559	0.0417
	0.99	$\hat{h}_{HM}, \hat{d}$	202.6091	80.2523	71.0121	26.5730	29.5662	10.6636
		$\hat{h}_{AM}, \hat{d}$	178.1331	132.5749	59.5868	44.3322	24.3004	18.2431
		$\hat{h}_{GM}, \hat{d}$	181.0734	132.4458	60.8711	44.2476	24.8943	18.1911
	0.8	$\hat{h}_{HM}, \hat{d}$	0.1345	0.0740	0.1304	0.0695	0.0101	0.0052
		$\hat{h}_{AM}, \hat{d}$	0.1295	0.0994	0.1261	0.0936	0.0109	0.0089
		$\hat{h}_{GM}, \hat{d}$	0.1284	0.0978	0.1250	0.0920	0.0092	0.0071
	0.9	$\hat{h}_{HM}, \hat{d}$	0.3269	0.1733	0.0468	0.0243	0.0182	0.0112
		$\hat{h}_{AM}, \hat{d}$	0.3295	0.2580	0.0500	0.0362	0.0231	0.0182
100		$\hat{h}_{GM}, \hat{d}$	0.3276	0.2565	0.0475	0.0346	0.0196	0.0162
	0.95	$\hat{h}_{HM}, \hat{d}$	2.4620	1.0325	0.4013	0.1384	0.1428	0.0478
		$\hat{h}_{AM}, \hat{d}$	2.4436	1.7542	0.3811	0.2224	0.1373	0.0701
		$\hat{h}_{GM}, \hat{d}$	2.4549	1.7524	0.3859	0.2217	0.1377	0.0698
	0.99	$\hat{h}_{HM}, \hat{d}$	1940.5680	694.8267	570.7769	182.9764	241.1576	69.0445
		$\hat{h}_{AM}, \hat{d}$	1697.5980	1141.1820	475.4377	310.2184	196.0655	119.0545
		$\hat{h}_{GM}, \hat{d}$	1728.0720	1139.8680	486.9897	309.5811	201.7439	118.7153

4. Results and discussion

The simulated MSE results for the estimators under study (i.e. the ATPE and AATPE) are presented in Tables 1–3 for the situation when the errors are distributed as the standardized normal, $t(6)$ and $F(4, 16)$, respectively. Estimators' performance is evaluated by varying the following factors: (i) degree of heteroscedasticity (δ), (ii) the level of collinearity (λ), (iii) the sample size (T) (iv) probability distribution of the disturbance term and (v) the estimators of biasing parameters h and d. Firstly, we consider the case where error term distributed as $N(0,\, 1)$ for which the results are given in Table 1. For homoscedastic error term ($\delta = 1$), the

ATPE yields lower MSE in comparison with AATPE for all levels of collinearity and for all sample sizes. The increase in level of collinearity causes the increase in MSE of the both estimator under consideration while MSE of both estimators decreases when the sample size increases. In case of mild heteroscedasticity ($\delta \approx 4$), a slight increase is observed in MSE of the both estimators, however, MSE of our proposed AATPE remains less than that of the ATPE for all cases of level of collinearity as well as for all cases of biasing parameters h and d. MSE of the both estimators increases more as the degree of heteroscedasticity increases.

For the case of $\delta \approx 36$ (moderate heteroscedasticity), more increase is observed in MSE of the both estimators. Though, our proposed AATPE remains efficient, in terms of smaller MSE, in comparison with the ATPE for all levels of collinearity. Nearly identical behavior is observed for $\delta \approx 100$ (the case of severe heteroscedasticity). The AATPE is significantly more efficient than the ATPE in situations where the degree of heteroscedasticity and the level of collinearity are high (i.e. $\delta \approx 100$ and $\lambda = 0.99$). Overall, we can say that the ATPE's performance becomes worst under heteroscedasticity. In contrast, our proposed AATPE demonstrates better performance as compare to the ATPE, in the sense of MSE, when the DLM is affected by heteroscedasticity of unknown form. When estimators of biasing parameters k and d are compared, we note that the ATPE gives larger MSE when pair of biasing estimators $(\hat{h}_{HM}, \hat{d}_{opt})$ is used while, in the case of AATPE, the $(\hat{h}_{HM}, \hat{d}_{opt})$ gives lower MSE as compared to the cases of $(\hat{h}_{AM}, \hat{d}_{opt})$ and $(\hat{h}_{GM}, \hat{d}_{opt})$.

The gain in efficiency is larger when the AATPE with $(\hat{h}_{HM}, \hat{d}_{opt})$ is used. For example, when $\delta \approx 36$, $\lambda = 0.80$ and using $(\hat{h}_{HM}, \hat{d}_{opt})$, the gain in efficiency for the cases of $T = 30, 60$ and 120, is $111.94\%, 112.28\%$ and 84.43%, respectively. The efficiency gains for the same level of collinearity and degree of heteroscedasticity are 59.63%, 38.76%, and 15.26% when $\hat{h}_{AM}$ and $\hat{d}_{opt}$ are used for biasing parameters. Similarly, the efficiency gains are 65.39%, 53.16%, and 27.57% for $\hat{h}_{GM}$ and $\hat{d}_{opt}$. Thus, we can conclude that the AATPE with the biasing parameter estimators $\hat{h}_{HM}$ and $\hat{d}_{opt}$ demonstrates exceptional performance.

Almost same behavior of is observed for the estimators when the error follows $t(6)$ and $F(4, 16)$ probability distribution (see, Tables 2 and 3). The same outstanding performance of the AATPE can be seen as for normal errors. Although, he ATPE becomes increasingly inefficient with heteroscedastic and non-normal error terms, particularly in cases of standardized distribution $F(4, 16)$.

5. Conclusion

It is common to use the AE for DLM estimation to avoid multicollinearity issue associated with the Almon technique. Özbay and Kaçiranlar (2017) proposed the ATPE which shows attractive performance over the usual AE. The situation worsens for the ATPE, which becomes inefficient when the error term of the DLM is affected by heteroscedasticity of an unknown form. This study is intended to encounter this problem. This study aims to propose an adaptive variant of the ATE (AATPE), which demonstrates superior performance over the ATPE when heteroscedastic errors are present. The performance of the proposed estimator is assessed through an extensive simulation framework, considering various factors such as the the level of collinearity, degree of heteroscedasticity, and the probability distribution of the error term. Our simulation results demonstrate that the proposed AATPE is significantly more efficient than the ATPE, even in the existence of severe multicollinearity and heteroscedasticity.

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