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Bayesian Rating using Glenn-David Paired Comparison Model with Non-Informative and Informative Priors

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Abstract

In the technique of paired comparisons, objects are ranked since individual judgment. We use it when quantifiable measurement is not viable or impractical. In this study, the Glenn-David PC model under Bayesian framework is used to establish the rating of five brands of cold drinks. Bayesian analysis has been made using non-informative and informative priors. The posterior means are considered for the preference behavior of the cold drink brands. The predictive probabilities for a single future paired comparison of cold drink brands are also found. The posterior probabilities of the hypotheses for comparison of parameters for any two cold drink brands are obtained. Also, the preference probabilities for paired comparison are determined. The results obtained through Uniform and Normal-Gamma priors are compared. It is observed that similar results and same rankings for the cold drinks brands are achieved. The appropriateness of the model is tested by chi-squared statistic. All computations are made in SAS package by designing the programs/codes.

Keywords: Bayesian rating, Non-informative priors, Glenn-David model, Paired comparison models.

1. Introduction

The key characteristic of Bayesian statistics is the existence of rules that allow researchers to repair existing beliefs after integrating new details. The comparison of two or more sets of policy making results can also be made under Bayesian approach by applying diverse techniques. One such technique is known as paired comparison (PC). This is seen that there are several goods and/or services in our society that are being compared with each other having minor differences; thus, it is difficult for the respondents to differentiate between them. The method of PC is commonly used to eliminate such complexities. This technique is very trustworthy and candid in the rating of under-discussion treatments. In general, the items are presented in couples to the assessors, who simply state the preference of one treatment over the other. The significance of the existing PC techniques is highlighted in several studies available in the literature. Glenn and David (1960) have modified Thurstone-Mosteller method of preference-ordering to cover cases in which ties are allowed. It is consummated by assuming that when the difference between a judge's responses to two stimuli under comparison occur below

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a certain verge a tie will be acknowledged. This threshold and the mean stimulus responses are estimated by least square. Tutz and Schauberger (2015) suggested a generalization of PC model and use it for the assessment of game competitions. Further explanatory variables are included in the extension of the model. The significance of methods was explained by investigating the performance and its dependence on the budget for football teams of the German Bundesliga. Csató (2015) analyzed the least squares ranking technique for generalized contests with probable missing and manifold pairwise comparisons. He had explored the relation amongst this technique and other way demarcated for ranking the nodes in a digraph. Altaf et al. (2013) conducted analysis of the Rao Kupper Model having order effect. The analysis was done via non-informative priors. The joint and marginal posterior distributions were obtained and estimates of the parameters were computed.

Aslam and Shah (2015) made comparative analysis of non-informative and informative priors, which were taken for the parameters of Bradley-Terry model. Lindley Shannon information was used for the comparison. Liu and Shih (2016) proposed a fresh decision tree technique for the analysis of PC data. It was shown that some covariates were the base of the preference patterns of the subjects.

Aslam and Kifayat (2018) analyzed Raleigh PC model under loss functions using informative priors; Conjugate and Dirichlet. The analysis of the study was based on real-life data of cigarette brands. Three loss functions were also practiced for estimation of the model parameters. They observed that the Bayes estimators (BEs) under Squared Error Loss Function had the overall minimum risk, as compared to other two loss functions for both the informative priors. Beaudoin and Swartz (2018) presented an innovative ranking scheme which was simple in concept. The study showed that in case of the absence or uncertainty of direct preferences, indirect comparisons could be made to obtain the favorites. The computational problem in determination of an ideal ranking was also discussed.

Sindhu et al. (2019) derived the probability density function of mixture for inverse Maxwell density. Simulation and real-life study had been conducted to inspect the comparative performance of BEs and the problems of employing priors and loss functions at different sample sizes. (Aslam & Cheema, 2020) conducted Bayesian analysis of 3-component mixture of exponentiated Weibull distribution under type-I right censoring scheme through noninformative priors. BEs and posterior risks (PRs) were found using loss functions and noninformative priors. BEs and PRs were observed as the function of test termination time. Awan and Aslam (2020) developed a PC model using Weibull distribution. The new model was further examined under Bayesian context through loss functions and informative priors. The posterior estimates were obtained to judge the rating of different social media applications. The developed model was found credible by the goodness of fit test. Varin and Firth (2024) discussed the estimation of paired comparison models by using ridge penalty. A modified approach was established that integrates empirical Bayes and composite likelihoods, eliminating the requirement to re-fit the model and offering a convenient alternative to cross validating the ridge tuning parameter. It was shown through simulation studies that the predictive accuracy of the new approach is significantly improved as compared to ordinary maximum likelihood.

Tian et al. (2024) presented a spectral ranker, named Kernel Rank Centrality, which proposed the ranking of the items based on pairwise comparison over time. The ranker utilizes kernel smoothing within the Bradley–Terry model and incorporates a Markov chain framework. Unlike the maximum likelihood method, the spectral ranker is nonparametric, requires fewer model assumptions and computations, and enables real-time ranking.

In this study, Glenn-David has been executed for PC using non-informative and informative priors to obtain the Bayesian rating of different cold drink brands. Comparative analysis has also been made for the results obtained through both kinds of priors. It is shown that a similar pattern of ranking has been achieved in both the cases.

The sequence of the article is Glenn-David PC model is presented in section 2. Section 3 covers selection of priors, Bayesian analysis of the subject model using Uniform and Normal-Gamma priors, preference, predictive and posterior probabilities is presented in section 4. Appropriateness of the model is displayed in section 5. Section 6 shows the conclusion of the study.

2. The Glenn-David PC model

Let $T_1, T_2, T_3, \ldots, T_m$ be m treatments, which are to be covered under pairwise comparison. There are $m(m-1)/2$ possible pairs, where each pair (T_i, T_j), $(i < j; 1 < i, j < m)$ is ranked r_{ij} times with $\theta_1, \theta_2, \theta_3, \ldots, \theta_m$ be the true preference of these m treatments, such that the judge when confronted with the treatment T_i, responds with a latent variable X_i. T_i is preferred to T_j if $X_i > X_j$.

Thurstone (1927) introduced the concept of paired comparison experiment for the subjective study of relative strength of stimuli. The most general model defined as Thurstonian model on m stimuli is based on multivariate distribution of $(X_1, X_2, X_3, \ldots, X_m)$ where stimuli are presented two at a time and pair wise choice probabilities take the form as

$$\Pr(X_i > X_j) = \frac{1}{\sqrt{2\pi(\sigma_i^2 + \sigma_j^2 - 2\sigma_{ij})}} \int_{\frac{\theta_i - \theta_j}{\sqrt{\sigma_i^2 + \sigma_j^2 - 2\sigma_{ij}}}}^{\infty} \exp\left(-\frac{y^2}{2}\right) dy, \tag{1}$$

where, θ_i and θ_j are true treatment locations on the sensation range. Thurstone assumes no correlation between the treatment stimuli. The full blown Thurstone model has too many parameters in the form of means, variances and covariance. Therefore, simplifying assumptions are applied to this model. Mosteller (1951) uses the same model assuming equal correlations but with no change in method. He summarizes the following principles:

1. There is a set of stimuli, which can be located on a subjective scale.
2. Each stimulus, when presented to an individual, gives rise to a sensation in the individual.
3. The distribution of sensation from a particular stimulus for a population of individuals is normal.

4. It is possible for paired sensations to be correlated.

Thus, the model may be summarized as

$$\Pr(X_i > X_j) = \frac{1}{\sqrt{2\pi}} \int_{-(\theta_i-\theta_j)}^{\infty} \exp\left(-\frac{y^2}{2}\right) dy, \tag{2}$$

where, θ_i and θ_j are true treatment locations on the sensation continuum and $\Pr(X_i > X_j)$ is the probability that treatment i is preferred to treatment j, usually denoted by ψ_{ij}. However, both Thurstone and Mosteller assumed all the differences are perceptible, even if they are very small and hence, they prohibit the ties.

Glenn and David (1960) have used a modified version of Thustone-Mosteller model to develop a method of estimating the relative strength of treatment stimulus which makes provision for tied observations. According to Thurstone-Mosteller model, we may write the two probabilities of preferences of T_i to T_j and T_j to T_i as $\psi_{ij} = \Pr(X_i > X_j) = F(\theta_i - \theta_j)$ and $\psi_{ji} = \Pr(X_j > X_i) = F(\theta_j - \theta_i)$ respectively, where $F(\theta_i - \theta_j)$ and $F(\theta_j - \theta_i)$ are the standard normal cumulative distribution functions.

Glenn and David suggest the third type of judgement i.e. no preference. They propose that if the difference between the two sensations due to two treatments $(X_i - X_j)$ lies below a certain threshold, then the panelist can declare a tie i.e. unable to detect any preference. We may say that there exists an interval of length $2\delta(-\delta, \delta)$, centered at the origin of the distribution of $(X_i - X_j)$, where the judge will declare a tie.

Now we denote the probability of T_i prefered to T_j as $\psi_{i.ij}$

$$\psi_{i.ij} = \Pr\left(X_i - X_j > \delta \mid \theta_i,\, \theta_j\right) = \frac{1}{\sqrt{2\pi}} \int_{-(\theta_i-\theta_j)+\delta}^{\infty} \exp\left(-\frac{y^2}{2}\right) dy \tag{3}$$

$$\psi_{i.ij} = \Phi(\theta_i - \theta_j - \delta),$$

where, $i \neq j$; $i,\, j = 1,\, 2,\, \ldots,\, m$, and Φ is the standard normal cumulative distribution function. The probability of preferring T_j to T_i is denoted by $\psi_{j.ij}$ and is defined as

$$\psi_{j.ij} = \Pr(X_j - X_i > \delta \mid \theta_i,\, \theta_j) = \frac{1}{\sqrt{2\pi}} \int_{-(\theta_j-\theta_i)+\delta}^{\infty} \exp\left(-\frac{y^2}{2}\right) dy \tag{4}$$

$$\psi_{j.ij} = \Phi(\theta_j - \theta_i - \delta),$$

where, $i \neq j$; $i,\, j = 1,\, 2,\, \ldots,\, m$. Consider the probability of declaring a tie *i.e.* $X_i - X_j$ lies between $(-\delta,\, \delta)$ is denoted by $\psi_{o.ij}$

$$\psi_{o.ij} = \Pr(\|X_i - X_j\| < \delta \mid \theta_i,\, \theta_j) = \frac{1}{\sqrt{2\pi}} \int_{-(\theta_i-\theta_j)-\delta}^{-(\theta_i-\theta_j)+\delta} \exp\left(-\frac{y^2}{2}\right) dy \tag{5}$$

$$\psi_{o.ij} = \Phi(\theta_j - \theta_i + \delta) - \Phi(\theta_j - \theta_i - \delta),$$

where, $i \neq j$; $i,\, j = 1,\, 2,\, \ldots,\, m$.

3. Likelihood function for the Glenn-David PC model

The probability of the perceived outcome in the kth repetition of the pair $(A_i,\, A_j)$ is

$$P_{ijk} = \{\Phi(\theta_i – \theta_j - \delta)\}^{n_{i.ijk}}\{\Phi(\theta_j - \theta_i - \delta)\}^{n_{j.ijk}}\{\Phi(\theta_i - \theta_j + \delta) - \Phi(\theta_i - \theta_j - \delta)\}^{n_{0.ijk}}$$

Therefore, the likelihood function for the Glenn and David model, starting with the predefined probabilities of the observed result $\mathbf{x}$ of the trial is

$$\begin{aligned} l(x;\theta_1,\,\theta_2,\,\ldots,\,\theta_m,\,\delta) &= \prod_{i<j=1}^{m}\prod_{k=1}^{r_{ij}} P_{ijk} \\ &= \prod_{i<j=1}^{m}\prod_{k=1}^{r_{ij}} \binom{r_{ij}}{n_{i.ij}\; n_{j.ij}\; n_{o.ij}} \{\Phi(\theta_i – \theta_j - \delta)\}^{n_{i.ijk}} \\ &\quad \{\Phi(\theta_j - \theta_i - \delta)\}^{n_{j.ijk}}\{\Phi(\theta_i - \theta_j + \delta) - \Phi(\theta_i - \theta_j - \delta)\}^{n_{0.ijk}} \\ &= \prod_{i<j=1}^{m} \binom{r_{ij}}{n_{i.ij}\; n_{j.ij}\; n_{o.ij}} \{\Phi(\theta_i – \theta_j - \delta)\}^{n_{i.ij}} \\ &\quad \{\Phi(\theta_j - \theta_i - \delta)\}^{n_{j.ij}}\{\Phi(\theta_i - \theta_j + \delta) - \Phi(\theta_i - \theta_j - \delta)\}^{n_{o.ij}} \end{aligned}$$

where, $-\infty < \theta_i < \infty$; $i = 1,\, 2,\, \ldots, m$ are the treatment parameters with $\sum_i^m \theta_i = 0$, and $\delta > 0$ is the tie. The constraint on treatment parameters, $\sum_i^m \theta_i = 0$, confirms that the parameters are quite well described. The notations used for the model are detailed in Table 1. Let $n_{i.ijk}$ and $n_{o.ijk}$ be the random variables related to the treatments' rank in the kth repetition of the treatment pair $(A_i,\, A_j)$, where, $1 \le i \ne j \le m$, and $k = 1,\, 2,\, 3,\, \ldots, r_{ij}$.

Table 1: Notations and their description.

Notations	**Description**
$n_{i.ijk}$	1 or 0 conforming by way of treatment A_i is chosen over treatment A_j or not chosen in the kth repetition of comparison.
$n_{o.ijk}$	1 or 0 corresponding as the treatment A_i is tied with treatment A_j or not in the kth repetition of comparison.
$n_{i.ij} = \sum_k n_{i.ijk}$	Numbers of times treatment A_i is preferred over treatment A_j.
$n_{o.ij} = \sum_k n_{0.ijk}$	Number of times treatment A_i and A_j are equally preferred.
r_{ij}	Number of times treatment A_i is compared with treatment A_j and $n_{i.ij} + n_{j.ij} + n_{o.ij} = r_{ji} = r_{ij}$. Moreover, $n_{o.ijk} + n_{i.ijk} + n_{j.ijk} = 1$.

4. The selection of prior distribution

4.1. Non-informative prior (Uniform distribution)

The uniform prior is assumed for the cold drink parameters, which is defined as

$p(\theta_1, \delta) \propto 1,$

where, $-\infty \leq \theta_1 \leq \infty$, and $\delta > 0$. By using the Uniform prior and the general form of the likelihood function of Glenn-David PC model, the joint posterior distribution is derived for 5 cold drink brands which is presented as

$$\begin{aligned} p\left(\theta_1, \theta_2, \theta_3, \theta_4, \delta \mid \mathbf{x}\right) = K_2^{-1} & \{\Phi(\theta_1-\theta_2-\delta)\}^{n_{1.12}}\{\Phi(\theta_2-\theta_1-\delta)\}^{n_{2.12}} \\ & \{\Phi(\theta_1-\theta_2+\delta)-\Phi(\theta_1-\theta_2-\delta)\}^{n_{0.12}} \\ & \{\Phi(\theta_1-\theta_3-\delta)\}^{n_{1.13}}\{\Phi(\theta_3-\theta_1-\delta)\}^{n_{3.13}} \\ & \{\Phi(\theta_1-\theta_3+\delta)-\phi(\theta_1-\theta_3-\delta)\}^{n_{0.13}} \\ & \{\Phi(\theta_2-\theta_3-\delta)\}^{n_{2.23}}\{\Phi(\theta_3-\theta_2-\delta)\}^{n_{3.23}} \\ & \{\Phi(\theta_2-\theta_3+\delta)-\Phi(\delta_2-\theta-\delta)\}^{n_{0.23}} \\ & \{\Phi(\theta_1-\theta_4-\delta)\}^{n_{1.14}}\{\Phi(\theta_4-\theta_1-\delta)\}^{n_{4.14}} \\ & \{\Phi(\theta_1-\theta_4+\delta)-\Phi(\theta_1-\theta_4-\delta)\}^{n_{0.14}} \\ & \{\Phi(\theta_1-\theta-\delta)\}^{n_{1.15}}\{\Phi(\theta_5-\theta_1+\delta)\}^{n_{5.15}} \\ & \{\Phi(\theta_1-\theta_5+\delta)-\Phi(\theta_1-\theta_5-\delta)\}^{n_{0.15}}. \end{aligned} \tag{6}$$

Here K_2 is defined as the normalizing constant and is given by

$$\begin{aligned} K_2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty} & \{\Phi(\theta_1-\theta_2-\delta)\}^{n_{1.12}}\{\Phi(\theta_2-\theta_1-\delta)\}^{n_{2.12}} \\ & \{\Phi(\theta_1-\theta_2+\delta)-\Phi(\theta_1-\theta_2-\delta)\}^{n_{0.12}} \\ & \{\Phi(\theta_1-\theta_3-\delta)\}^{n_{1.13}}\{\Phi(\theta_3-\theta_1-\delta)\}^{n_{3.13}} \\ & \{\Phi(\theta_1-\theta_3+\delta)-\phi(\theta_1-\theta_3-\delta)\}^{n_{0.13}} \\ & \{\Phi(\theta_2-\theta_3-\delta)\}^{n_{2.23}}\{\Phi(\theta_3-\theta_2-\delta)\}^{n_{3.23}} \\ & \{\Phi(\theta_2-\theta_3+\delta)-\Phi(\delta_2-\theta-\delta)\}^{n_{0.23}} \\ & \{\Phi(\theta_1-\theta_4-\delta)\}^{n_{1.14}}\{\Phi(\theta_4-\theta_1-\delta)\}^{n_{4.14}} \\ & \{\Phi(\theta_1-\theta_4+\delta)-\Phi(\theta_1-\theta_4-\delta)\}^{n_{0.14}} \\ & \{\Phi(\theta_1-\theta-\delta)\}^{n_{1.15}}\{\Phi(\theta_5-\theta_1+\delta)\}^{n_{5.15}} \\ & \{\Phi(\theta_1-\theta_5+\delta)-\Phi(\theta_1-\theta_5-\delta)\}^{n_{0.15}}\mathrm{d}\delta\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2\,\mathrm{d}\theta_3\,\mathrm{d}\theta_4, \end{aligned}$$

and $\theta_5 = -\theta_1 - \theta_2 - \theta_3 - \theta_4$.

4.2. Informative prior (Normal-Gamma prior)

It is essential to take greater care while selecting priors. The reason to adopt care is because inappropriate choices for priors can lead to incorrect interpretations. Sometimes it is convenient to develop marginal prior for each parameter instead of developing a joint prior for all the parameters. Accordingly, the prior distributions for the cold drink parameters $\theta_1, \theta_2, \ldots, \theta_m$ and tie parameter δ are considered to be independent. The prior distributions are presumed conferring to the supports of the parameters for the model. With the experts' guidance, the Normal distribution is assumed as prior distribution for the cold drink parameters and the Gamma distribution is supposed for tie parameter. Such as

$$p(\theta_i) = \frac{1}{\sqrt{2\pi\sigma_i^2}} e^{-\frac{(\theta_i-\mu_i)^2}{2\sigma_i^2}}; \qquad -\infty \leq \mu_i \leq \infty,\ \sigma_i^2 > 0,\ i = 1,\ 2,\ \ldots, m,$$

and

$$p(\delta) = \frac{\beta^\alpha}{\Gamma\alpha} \delta^{\alpha-1} e^{-\delta\beta} \qquad \alpha,\ \beta > 0.$$

Therefore, the joint prior distribution of $\theta_1, \theta_2, \ldots, \theta_m$ and δ is Normal-Gamma (N-G) distribution which is defined as

$$p(\theta_i, \delta) = \prod_{i=1}^{m} \frac{1}{\sqrt{2\pi\sigma_i^2}} e^{-\frac{(\theta_i-\mu_i)^2}{2\sigma_i^2}} \frac{\beta^\alpha}{\Gamma\alpha} \delta^{\alpha-1} e^{-\delta\beta}. \tag{7}$$

By using density kernel in Eq. (5) can be written as

$$p(\theta_i, \delta) \propto \prod_{i=1}^{m} e^{-\frac{(\theta_i-\mu_i)^2}{2\sigma_i^2}} \delta^{\alpha-1} e^{-\delta\beta}$$

where, $i = 1, 2, \ldots, m$ and μ_i, σ_i^2, α and β are hyperparameters. The joint posterior distribution using N-G prior of the cold drink parameters $\theta_1, \theta_2, \ldots, \theta_m$ and δ given data $\mathbf{x}$ is

$$p(\theta_i, \delta) \propto \delta^{\alpha-1} e^{-\delta\beta} \prod_{i<j=1}^{m} e^{-\frac{(\theta_i-\mu_i)^2}{2\sigma_i^2}} \{\Phi(\theta_i - \theta_j - \delta)\}^{n_{i.ij}}$$
$$\{\Phi(\theta_i - \theta_j + \delta) - \Phi(\theta_i - \theta_j - \delta)\}^{n_{o.ij}} \{\Phi(\theta_j - \theta_i - \delta)\}^{n_{j.ij}}, \tag{8}$$

where, $-\infty < \theta_i < \infty$; $i = 1, 2, \ldots, m$ and δ ,with $\sum_{i=1}^{m} \theta_i = 0$.

Now, by taking an experts' opinion we have selected two different sets of assumed values of the hyper-parameters. These sets of values are given in Table 2.

Table 2: Assumed values of hyper-parameters.

Set A		Set B	
$\mu_1 = 2$	$\sigma_1^2 = 1$	$\mu_1 = 3$	$\sigma_1^2 = 1$
$\mu_2 = 2$	$\sigma_2^2 = 1$	$\mu_2 = 4$	$\sigma_2^2 = 2$
$\mu_3 = 2$	$\sigma_3^2 = 1$	$\mu_3 = 2$	$\sigma_3^2 = 1$
$\mu_4 = 2$	$\sigma_4^2 = 1$	$\mu_4 = 3$	$\sigma_4^2 = 1$
$\alpha = 2$	$\beta = 1$	$\alpha = 3$	$\beta = 1$

Table 3: Real life data.

Cold Drink Brands	Pair (i, j)	$n_{i.ij}$	$n_{j.ij}$	$n_{o.ij}$	Total
(PE, CO)	(1, 2)	8	19	3	30
(PE, SU)	(1, 3)	9	20	1	30
(PE, SP)	(1, 4)	13	15	2	30
(PE, DU)	(1, 5)	20	8	2	30
(CO, SU)	(2, 3)	11	18	1	30
(CO, SP)	(2, 4)	16	13	1	30
(CO, DU)	(2, 5)	21	8	1	30
(SU, SP)	(3, 4)	14	12	4	30
(SU, DU)	(3, 5)	24	3	3	30
(SP, DU)	(4, 5)	18	9	3	30

5. Analysis of the model

To conduct the analysis under the subject model, five brands of cold drinks namely: PEPSI (PE) and CoCa Cola (CO), 7Up (SU), Sprite (SP) and Dew (DU) are considered. So, in this case, we have 5 parameters θ_1, θ_2, θ_3, θ_4 and θ_5 which are used for the decision of rating of the cold drinks being compared. Tie parameter is represented by δ. Real life data from 30 students at Punjab College Rawalpindi is collected. Students were given the choice to tag the preferred cold drink or to declare tie amongst the two brands. The data received from the students is appended below in Table 3.

5.1. Graphical representations of marginal posterior densities

The graphical representation of the posterior densities of the cold drink parameters are presented in Figures 1 to 6. It is apparent that the graphs obtained through N-G and UP, show the similar tendency.

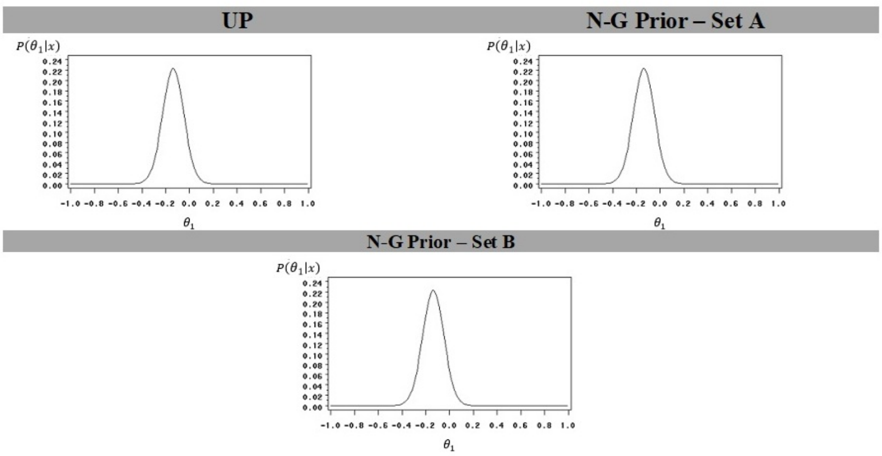

Figure 1: Posterior distribution of θ_1 using UP and N-G priors.

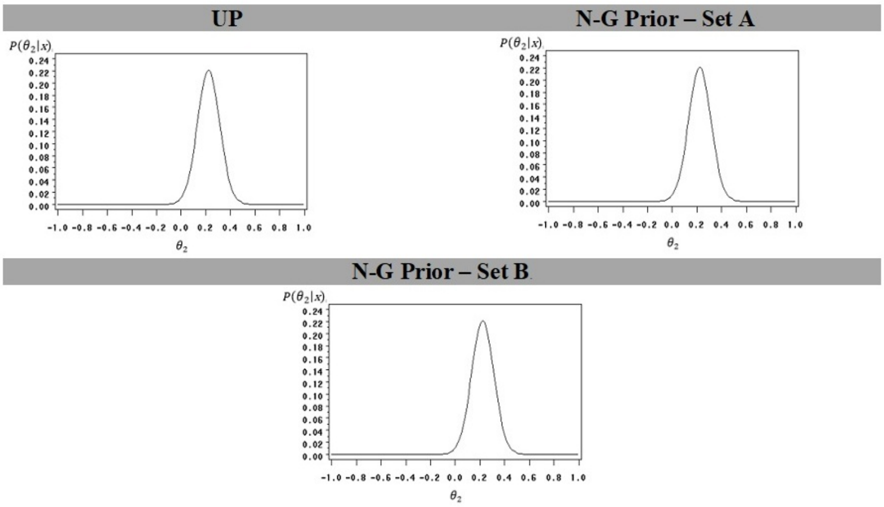

Figure 2: Posterior distribution of θ_2 using UP and N-G priors.

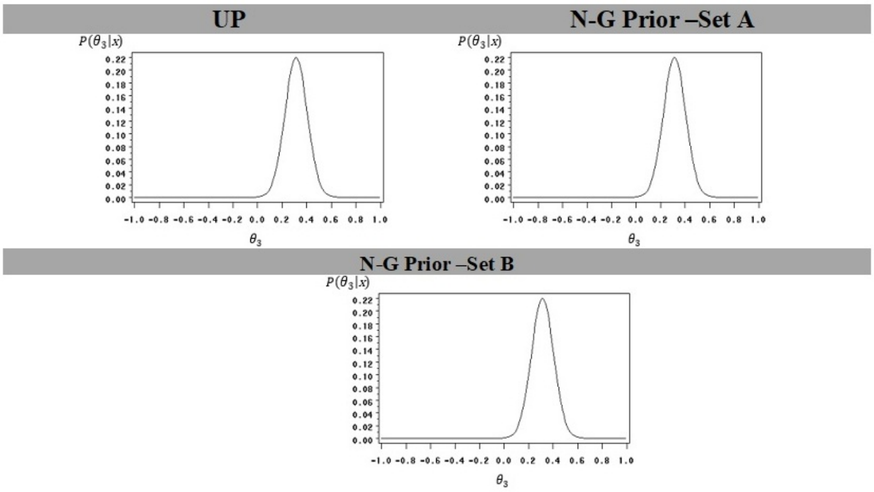

Figure 3: Posterior distribution of θ_3 using UP and N-G priors.

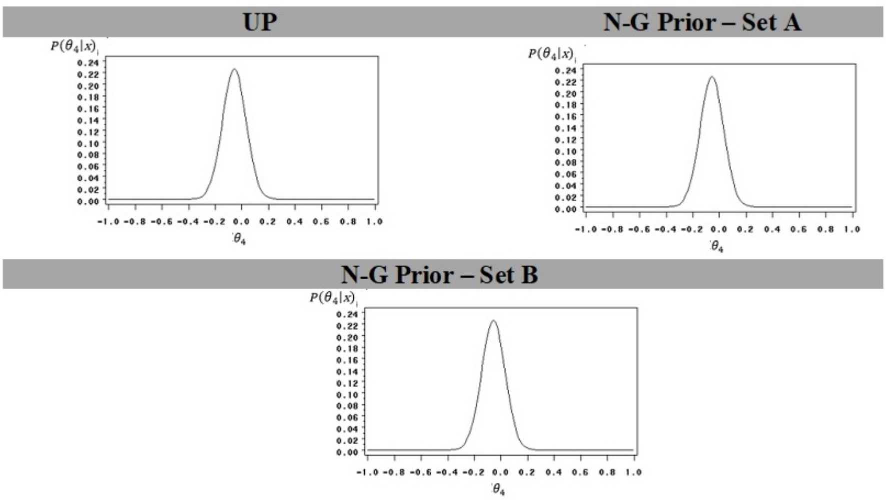

Figure 4: Posterior distribution of θ_4 using UP and N-G priors.

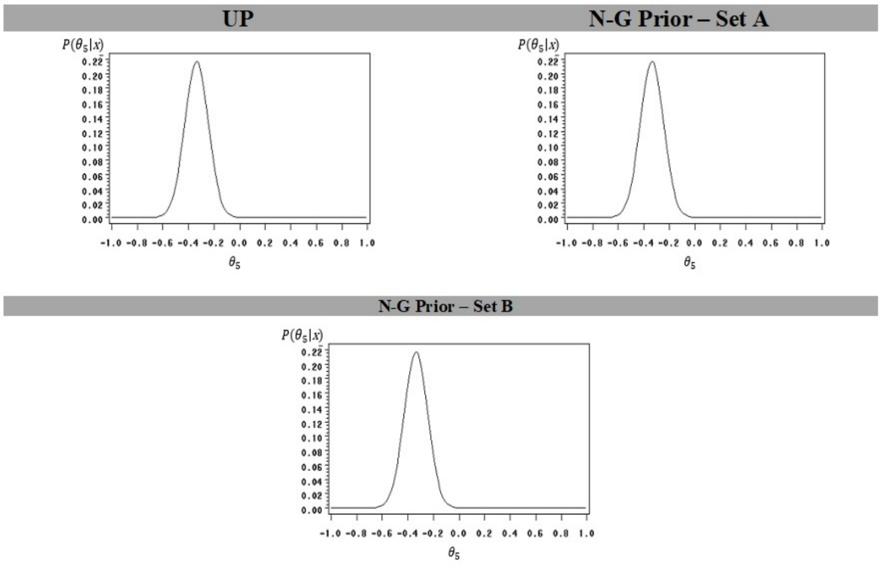

Figure 5: Posterior distribution of θ_5 using UP and N-G priors.

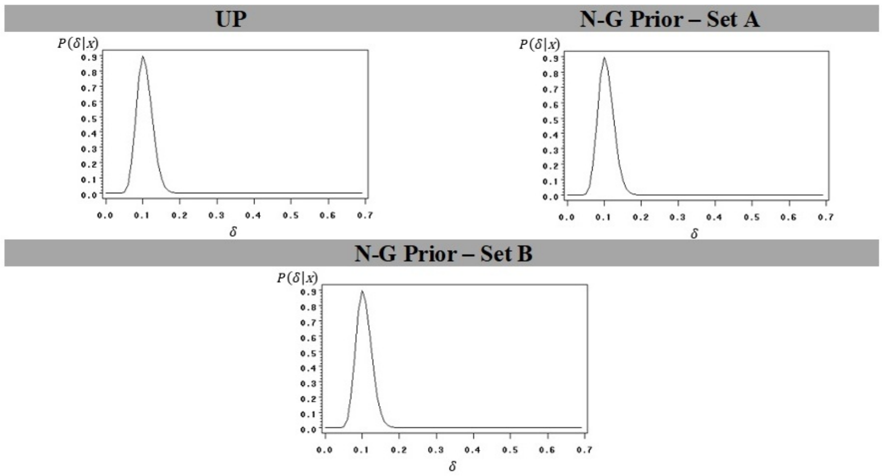

Figure 6: Posterior distribution of δ using UP and N-G priors.

5.2. Posterior estimates using N-G prior

The posterior estimates for the cold drink parameters are obtained in the form of means in Table 4 which are obtained by designing the program in the SAS package. The posterior estimates attained by using informative (N-G) and non-informative (UP) Priors are in concurrence up to one decimal place and show same liking arrangement. It is perceived that the value of the mean of the parameter for SU is the leading value amongst the mean values of the other parameters. So, it is established that SU is the best one. Whereas the smallest posterior mean is of the parameter for DU. According to the obtained values of the posterior means, the rating of the five cold drink brands is as follows.

$$\text{SU} \rightarrow \text{CO} \rightarrow \text{SP} \rightarrow \text{PE} \rightarrow \text{DU}.$$

Table 4: Posterior means.

Cold Drink Brands	Parameters	Posterior means		
		UP	N-G Set A	N-G Set B
PE	θ_1	-0.1489	-0.1429	-0.1372
CO	θ_2	0.2206	0.2237	0.2208
SU	θ_3	0.3159	0.3182	0.3142
SP	θ_4	-0.0675	-0.0623	-0.0568
DU	θ_5	-0.3201	-0.3367	-0.3410
TIE parameter	δ	0.1009	0.1027	0.1046

5.3. Preference probabilities using N-G prior

The preference probabilities are calculated by substituting the posterior estimates in the model. These probabilities are listed in Table 5. It is evident from above table of preference probabilities that the results for five cold drink brands using informative (N-G) and non-informative (UP) Priors are obviously close to each other. Therefore, we obtained the same preference pattern for both the priors. For instance, by using the N-G prior with both the sets of assumed values of hyper-parameters, it is observed that preference probability of PE to CO is 0.3192 (set A), 0.3228 (set B) and probability of preferring CO to PE is 0.6026 (set A) and 0.5987 (set B). So, CO is preferred over PE. Likewise, interpretation of remaining pairs can be made in similar fashion.

Table 5: Preference probabilities using informative and non-informative priors.

Cold Drink Brands	Pair (i, j)	$\psi_{i.ij}$			$\psi_{j.ij}$			$\psi_{o.ij}$		
		UP	N-G Set A	N-G Set B	UP	N-G Set A	N-G Set B	UP	N-G Set A	N-G Set B
(PE, CO)	(1, 2)	0.3191	0.3192	0.3228	0.6064	0.6026	0.5987	0.0745	0.0783	0.0785
(PE, SU)	(1, 3)	0.2843	0.2877	0.2877	0.6406	0.6406	0.6368	0.0751	0.0717	0.0754
(PE, SP)	(1, 4)	0.4286	0.4286	0.4247	0.4920	0.5080	0.5080	0.0794	0.0634	0.0674
(PE, DU)	(1, 5)	0.5279	0.5359	0.5398	0.3936	0.3821	0.3745	0.0785	0.0821	0.0857
(CO,SU)	(2, 3)	0.4207	0.4207	0.4129	0.4920	0.5080	0.5080	0.0873	0.0713	0.0791
(CO, SP)	(2, 4)	0.5754	0.5714	0.5675	0.3483	0.3483	0.3520	0.0763	0.0803	0.0805
(CO, DU)	(2, 5)	0.6700	0.6772	0.6772	0.2611	0.2546	0.2514	0.0689	0.0681	0.0713
(SU, SP)	(3, 4)	0.6103	0.6103	0.6064	0.3156	0.3156	0.3121	0.0741	0.0741	0.0815
(SU, DU)	(3, 5)	0.7054	0.7088	0.7088	0.2295	0.2236	0.2236	0.0651	0.0675	0.0675
(SP, DU)	(4, 5)	0.5596	0.5675	0.5714	0.3632	0.3520	0.3483	0.0772	0.0805	0.0803

5.4. Bayesian testing of hypotheses using N-G prior

In Bayesian analysis, to make decision between two hypotheses H_0 and H_1 is theoretically straightforward. Following two hypotheses reflected the comparison of θ_i and θ_j

$$H_{ij} : \theta_i \geq \theta_j \quad \text{versus} \quad H_{ji} : \theta_j > \theta_i.$$

Table 6: Predictive probabilities using informative and non-informative priors.

Drink Brands	Pair (i, j)	$P_{i.ij}$			$P_{j.ij}$			$P_{o.ij}$		
		UP	N-G Set A	N-G Set B	UP	N-G Set A	N-G Set B	UP	N-G Set A	N-G Set B
(PE, CO)	(1, 2)	0.3207	0.3212	0.3235	0.6048	0.6030	0.5990	0.0745	0.0758	0.0775
(PE, SU)	(1, 3)	0.2877	0.2884	0.2911	0.6406	0.6386	0.6343	0.0717	0.0730	0.0747
(PE, SP)	(1, 4)	0.4284	0.4280	0.4273	0.4922	0.4913	0.4905	0.0794	0.0807	0.0823
(PE, DU)	(1, 5)	0.5277	0.5358	0.5391	0.3938	0.3847	0.3801	0.0785	0.0795	0.0808
(CO, SU)	(2, 3)	0.4229	0.4226	0.4223	0.4978	0.4968	0.4956	0.0793	0.0806	0.0821
(CO, SP)	(2, 4)	0.5734	0.5720	0.5680	0.3501	0.3502	0.3526	0.0765	0.0778	0.0795
(CO, DU)	(2, 5)	0.6683	0.6746	0.6745	0.2627	0.2560	0.2549	0.0690	0.0694	0.0707
(SU, SP)	(3, 4)	0.6101	0.6083	0.6040	0.3158	0.3162	0.3189	0.0741	0.0755	0.0771
(SU, DU)	(3, 5)	0.7017	0.7075	0.7070	0.2329	0.2269	0.2261	0.0654	0.0657	0.0669
(SP, DU)	(4, 5)	0.5597	0.5674	0.5706	0.3632	0.3546	0.3501	0.0771	0.0781	0.0793

For testing of the hypotheses, we follow the rule, confirmed by Aslam (1996) which is formulated as:
Suppose $t = \min(p_{ij}, q_{ij})$
H_{ij} is accepted when q_{ij} is small.
H_{ji} is accepted when p_{ij} is small.
If $t > 0.1$ then the testimony is not decisive.
The posterior probability of H_{ij} is

$$p_{ij} = \Pr(\theta_i > \theta_j) = \Pr(\theta_i > -\theta_j) = \Pr(\theta_i > 0 \mid \mathbf{x}), = \int_0^\infty \int_0^\infty p(\theta_i, \theta \mid \mathbf{x})\, d\delta\, d\theta, \quad (9)$$

where, $p(\theta_i, \delta \mid \mathbf{x})$ is representing the joint posterior distributions, which are defined in Eqs. (4) and (6). The posterior probability for H_{ji}, can be obtained as $q_{ij} = 1 - p_{ij}$. Table 7 presents the required posterior probabilities under the aforementioned hypotheses. By comparing the posterior probabilities, obtained by

Table 7: Posterior probabilities using informative and non-informative priors.

Drink Brands	Hypotheses	Posterior probabilities			Hypotheses	Posterior probabilities		
		UP	N-G Set A	N-G Set B		UP	N-G Set A	N-G Set B
(PE, CO)	$H_{12} : \theta_1 > \theta_2$	0.0074	0.0076	0.0090	$H_{21} : \theta_2 \geq \theta_1$	0.9926	0.9924	0.9910
(PE, SU)	$H_{13} : \theta_1 > \theta_3$	0.0009	0.0010	0.0013	$H_{31} : \theta_3 \geq \theta_1$	0.9991	0.9990	0.9987
(PE, SP)	$H_{14} : \theta_1 > \theta_4$	0.3436	0.3449	0.3451	$H_{41} : \theta_4 \geq \theta_1$	0.6564	0.6551	0.6549
(PE, DU)	$H_{15} : \theta_1 > \theta_5$	0.9131	0.9381	0.9465	$H_{51} : \theta_5 \geq \theta_1$	0.0869	0.0619	0.0535
(CO, SU)	$H_{23} : \theta_2 > \theta_3$	0.3107	0.3117	0.3146	$H_{32} : \theta_3 \geq \theta_2$	0.6893	0.6884	0.6854
(CO, SP)	$H_{24} : \theta_2 > \theta_4$	0.9868	0.9866	0.9844	$H_{42} : \theta_4 \geq \theta_2$	0.0132	0.0134	0.0156
(CO, DU)	$H_{25} : \theta_2 > \theta_5$	0.9999	0.9999	0.9999	$H_{52} : \theta_5 \geq \theta_2$	0.0001	0.0001	0.0001
(SU, SP)	$H_{34} : \theta_3 > \theta_4$	0.9981	0.9981	0.9977	$H_{43} : \theta_4 \geq \theta_3$	0.0019	0.0019	0.0023
(SU, DU)	$H_{35} : \theta_3 > \theta_5$	0.9999	0.9999	0.9999	$H_{53} : \theta_5 \geq \theta_3$	0.0001	0.0001	0.0001
(SP, DU)	$H_{45} : \theta_4 > \theta_5$	0.9751	0.9830	0.9858	$H_{54} : \theta_5 \geq \theta_4$	0.0249	0.0170	0.0142

using N-G and UP, it is found that the results are very near to each other. Hence, they can be interpreted in the same way. It is observed that the hypothesis H_{21} is accepted with high probabilities, i.e., $q_{12} = 0.9926$ (UP), $q_{12} = 0.9924$ (N-G-set A) and $q_{12} = 0.9910$ (N-G-set B). Likewise, $q_{13} = 0.9991$ (UP), $q_{13} = 0.9990$ (set A) and $q_{13} = 0.9987$ (set B) depict that the hypothesis H_{31} is accepted. Consequently,

Table 8: Observed and expected number of preferences with Uniform prior.

Drink Brands	Pairs $(i,\ j)$	$n_{i.ij}$	$\hat{n}_{i.ij}$	$n_{j.ij}$	$\hat{n}_{j.ij}$	$n_{o.ij}$	$\hat{n}_{o.ij}$
(PE, CO)	(1, 2)	8	9.57	19	18.19	3	2.24
(PE, SU)	(1, 3)	9	8.53	20	19.22	1	2.25
(PE, SP)	(1, 4)	13	12.86	15	14.76	2	2.38
(PE, DU)	(1, 5)	20	15.84	8	11.81	2	2.36
(CO, SU)	(2, 3)	11	12.62	18	14.76	1	2.62
(CO, SP)	(2, 4)	16	17.26	13	10.45	1	2.29
(CO, DU)	(2, 5)	21	20.10	8	7.83	1	2.07
(SU, SP)	(3 ,4)	14	18.31	12	9.47	4	2.22
(SU, DU)	(3, 5)	24	21.16	3	6.89	3	1.95
(SP, DU)	(4, 5)	18	16.79	9	10.90	3	2.32

it shows that SU is superior to PE. The posterior probabilities $p_{15} = 0.9131$ (UP) $p_{15} = 0.9381$ (set A) and $p_{15} = 0.9465$ (set B) show that the hypothesis H_{15} is accepted. Hence, it is concluded that PE is better than DU. Similarly, acceptance of the hypotheses H_{24}, H_{25}, H_{34}, H_{35} and H_{45} is observed from the above results. While the decisions are inconclusive for the hypothesis H_{23} and H_{14}.

5.5. Appropriateness of the model using uniform prior

For appropriateness of the model, chi-square statistic is used. Chi-square test is defined for the hypotheses:

H_0: The model is suitable for any value $\theta = \theta_0$.
H_1: The model is not suitable for any value of θ.

The obtained results are presented in Table 8. The value of χ^2 statistic is obtained to be 14.58 at 15 degrees of freedom, with $p-$value of 0.4816. It shows that there is no proof regarding the inappropriateness of the model.

5.6. Appropriateness of the model Using N-G prior for Set A

The chi^2 statistic is used to test the appropriateness of the Glenn-David model using N-G prior with Data Set A. The results are presented in Table 9. The χ^2-statistic yields a value of $\chi^2 = 13.070$ at 15 degrees of freedom, with $p-$value of 0.59649. Hence, there is no evidence that the model does not fit.

5.7. Appropriateness of the model using N-G prior for Set B

The results are presented in the Table 10. We get $\chi^2 = 12.773$. The $p-$value for this case is 0.61943. Again, there is no indication that the model does not fit.

!ht

Table 9: Observed and expected number of preferences for Data Set A

Drink Brands	Pairs $(i,\ j)$	$n_{i.ij}$	$\hat{n}_{i.ij}$	$n_{j.ij}$	$\hat{n}_{j.ij}$	$n_{o.ij}$	$\hat{n}_{o.ij}$
(PE, CO)	(1, 2)	8	9.58	19	18.08	3	2.35
(PE, SU)	(1, 3)	9	8.63	20	19.22	1	2.15
(PE, SP)	(1, 4)	13	12.86	15	15.24	2	1.90
(PE, DU)	(1, 5)	20	16.08	8	11.46	2	2.46
(CO, SU)	(2, 3)	11	12.62	18	15.24	1	2.14
(CO, SP)	(2, 4)	16	17.14	13	10.45	1	2.41
(CO, DU)	(2, 5)	21	20.32	8	7.64	1	2.04
(SU, SP)	(3, 4)	14	18.31	12	9.47	4	2.22
(SU, DU)	(3, 5)	24	21.27	3	6.71	3	2.03
(SP, DU)	(4, 5)	18	17.02	9	10.56	3	2.42

Table 10: Observed and expected number of preferences for Data Set B

Drink Brands	Pairs $(i,\ j)$	$n_{i.ij}$	$\hat{n}_{i.ij}$	$n_{j.ij}$	$\hat{n}_{j.ij}$	$n_{o.ij}$	$\hat{n}_{o.ij}$
(PE, CO)	(1, 2)	8	9.68	19	17.96	3	2.36
(PE, SU)	(1, 3)	9	8.63	20	19.10	1	2.26
(PE, SP)	(1, 4)	13	12.74	15	15.24	2	2.02
(PE, DU)	(1, 5)	20	16.19	8	11.23	2	2.57
(CO, SU)	(2, 3)	11	12.39	18	15.24	1	2.37
(CO, SP)	(2, 4)	16	17.02	13	10.56	1	2.42
(CO, DU)	(2, 5)	21	20.32	8	7.54	1	2.14
(SU, SP)	(3, 4)	14	18.19	12	9.36	4	2.45
(SU, DU)	(3, 5)	24	21.27	3	6.71	3	2.03
(SP, DU)	(4, 5)	18	17.14	9	10.45	3	2.41

6. Conclusion

The Bayesian study has been carried out to compare five cold drink brands using noninformative (UP) and Informative prior (Normal-Gamma prior). The joint informative prior distribution i.e., N-G prior is developed for five cold drink brands. By the experts' opinion two different sets of assumed values of hyperparameters have been engaged for the analysis under N-G prior. The comparison of the results of UP and N-G is briefly elucidated. The graphs of the marginal densities of the 5 cold drink parameters under GDM using UP and N-G prior portray similar outlook. No substantial discrepancy is perceived. Both type of priors marked the same rating to the given cold drink brands such that all noteworthy parameters have symmetrical graphs. Whereas, comparing the estimates of parameters for the cold drink brands which are attained using N-G and UP, it is observed that they are concurrent up to one fraction place and displayed the similar ratings by dint of both type of priors and for same feedback data set. Both type of priors explicit the same liking arrangement for five cold drink brands. It is identified that the parameter of SU has the greatest value among all the values of 5 parameters.

Consequently, it is concluded that SU has been considered as most favourite cold drink brand amongst five cold drink brands. Similarly, we have determined that CO has second, SP has third, PE has fourth and DU has fifth position with smallest value. The preference probabilities for cold drink brands obtained via N-G and UP, have been compared. It is observed that the results attained through both types of priors are obviously close to each other. The predictive probabilities have also been compared. The results are fairly analogous to each other. Therefore, it is interpreted that the ranking of five cold drink brands obtained by predictive probabilities of N-G prior is similar with the preference pattern obtained using predictive probabilities of UP. The posterior probabilities for testing the hypotheses of comparison of the parameters for the cold drink brands to examine that which cold drink parameters has superior probability when all have been compared pairwise. It is found that decisions of Hypotheses are same for both UP and N-G prior. As the hypotheses H_{15}, H_{21}, H_{24}, H_{31}, H_{25}, H_{34}, H_{35} and H_{45} are accepted with high probability. Whereas the decisions are inconclusive for the hypotheses H_{23} and H_{14} in case of both type of priors. The results of show that GDM using UP and N-G prior is a good, fitted model.

7. Proposal for future study

For further research it is recommended that

1. Instead of N-G prior some other informative priors can be used under Bayesian approach for paired comparison analysis.
2. The Bayesian study of the GDM with order effect can also be performed.
3. Another paired comparison model can be used and comparative analysis/study with GDM can be made.

Appendices

A. Program for Posterior Mean of θ_1 with JP for $m = 2$

```
DATA DD;
INPUT N012 N112 N212 DA DL;
CARDS;
3 8 19 0.0001 0.05;
PROC PRINT DATA=DD;  RUN;
DATA CC; SET DD;
DO T1=-4 TO 4-DL BY DL;
DO D=DL TO 4-DL BY DL;
T2=-T1;
PT=PROBNORM(2*T1-D)**N112*PROBNORM(-2*T1-D)**N212*
(PROBNORM(2*T1+D)-PROBNORM(2*T1-D))**N012;
/*EXPECTATION ON DATA*/
EN112=RR*(PROBNORM(2*T1-D));            RR=1;
EN212=RR*(PROBNORM(-2*T1-D));
EN012=RR*(PROBNORM(2*T1+D)-PROBNORM(2*T1-D));
```

```
/* LOG LIKELIHOOD FUNCTION i.e. LF */
LF=EN112*LOG(PROBNORM(2*T1-D))+EN212*LOG(PROBNORM(-2*T1-D))+
   EN012*LOG(PROBNORM(2*T1+D)-PROBNORM(2*T1-D));
/* NUMERICAL PARTIAL DIFFERENTIATION*/
INCT1=EN112*LOG(PROBNORM(2*(T1+DA)-D))+EN212*LOG(PROBNORM
   (-2*(T1+DA)-D))
+EN012*LOG(PROBNORM(2*(T1+DA)+D)-PROBNORM(2*(T1+DA)-D));
REDT1=EN112*LOG(PROBNORM(2*(T1-DA)-D))+EN212*LOG(PROBNORM
   (-2*(T1-DA)-D))
+EN012*LOG(PROBNORM(2*(T1-DA)+D)-PROBNORM(2*(T1-DA)-D));
INCD=EN112*LOG(PROBNORM(2*T1-(D+DA)))+EN212*LOG(PROBNORM(-2*
   T1-(D+DA)))
+EN012*LOG(PROBNORM(2*T1+(D+DA))-PROBNORM(2*T1-(D+DA)));
REDD=EN112*LOG(PROBNORM(2*T1-(D-DA)))+EN212*LOG(PROBNORM(-2*
   T1-(D-DA)))
+EN012*LOG(PROBNORM(2*T1+(D-DA))-PROBNORM(2*T1-(D-DA)));
INCT1D=EN112*LOG(PROBNORM(2*(T1+DA)-(D+DA)))+EN212*LOG(
   PROBNORM(-2*(T1+DA)-(D+DA)))+EN012*LOG(PROBNORM(2*(T1+DA)
   +(D+DA))-PROBNORM(2*(T1+DA)-(D+DA)));
REDT1D=EN112*LOG(PROBNORM(2*(T1-DA)-(D-DA)))+
EN212*LOG(PROBNORM(-2*(T1-DA)-(D-DA)))+EN012*
LOG(PROBNORM(2*(T1-DA)+(D-DA))-PROBNORM(2*(T1-DA)-(D-DA)));
DDT1=(INCT1+REDT1-2*LF)/DA**2;
           X1=DDT1;
DDD=(INCD+REDD-2*LF)/DA**2;
                   Y2=DDD;
PDT1D=(INCT1D+REDT1D-2*LF-DA**2*(DDT1+DDD))/(2*DA**2);  X2=
   PDT1D;
/* DETERMINANT OF FISHER INFORMATION MATRIX*/
DET=X1*Y2-X2**2;
JP4=SQRT(DET);
FUNC=(JP4*PROBNORM(2*T1-D)**N112*PROBNORM(-2*T1-D)**N212*(
   PROBNORM(2*T1+D)-PROBNORM(2*T1-D))**N012);
FUN1=DL**2*FUNC;
NC+FUN1;
END;END;RUN;
DATA DD1; SET CC;
DO T1=-4 TO 4-DL BY DL;
DO D=DL TO 4-DL BY DL;
T2=-T1;
PT=PROBNORM(2*T1-D)**N112*PROBNORM(-2*T1-D)**N212*(PROBNORM
   (2*T1+D)-PROBNORM(2*T1-D))**N012;
/*EXPECTATION ON DATA*/
EN112=RR*(PROBNORM(2*T1-D));             RR=1;
EN212=RR*(PROBNORM(-2*T1-D));
EN012=RR*(PROBNORM(2*T1+D)-PROBNORM(2*T1-D));
/* LOG LIKELIHOOD FUNCTION i.e. LF */
LF=EN112*LOG(PROBNORM(2*T1-D))+EN212*LOG(PROBNORM(-2*T1-D))+
   EN012*LOG(PROBNORM(2*T1+D)-PROBNORM(2*T1-D));
/* NUMERICAL PARTIAL DIFFERENTIATION*/
INCT1=EN112*LOG(PROBNORM(2*(T1+DA)-D))+
```

```
EN212*LOG(PROBNORM(-2*(T1+DA)-D))
+EN012*LOG(PROBNORM(2*(T1+DA)+D)-PROBNORM(2*(T1+DA)-D));
REDT1=EN112*LOG(PROBNORM(2*(T1-DA)-D))+
EN212*LOG(PROBNORM(-2*(T1-DA)-D))
+EN012*LOG(PROBNORM(2*(T1-DA)+D)-PROBNORM(2*(T1-DA)-D));
INCD=EN112*LOG(PROBNORM(2*T1-(D+DA)))+
EN212*LOG(PROBNORM(-2*T1-(D+DA)))
+EN012*LOG(PROBNORM(2*T1+(D+DA))-PROBNORM(2*T1-(D+DA)));
REDD=EN112*LOG(PROBNORM(2*T1-(D-DA)))+
EN212*LOG(PROBNORM(-2*T1-(D-DA)))
+EN012*LOG(PROBNORM(2*T1+(D-DA))-PROBNORM(2*T1-(D-DA)));
INCT1D=EN112*LOG(PROBNORM(2*(T1+DA)-(D+DA)))
+EN212*LOG(PROBNORM(-2*(T1+DA)-(D+DA)))+EN012*LOG(PROBNORM
   (2*(T1+DA)+(D+DA))-PROBNORM(2*(T1+DA)-(D+DA)));
REDT1D=EN112*LOG(PROBNORM(2*(T1-DA)-(D-DA)))+
EN212*LOG(PROBNORM(-2*(T1-DA)-(D-DA)))+EN012*LOG(PROBNORM
   (2*(T1-DA)+(D-DA))-PROBNORM(2*(T1-DA)-(D-DA)));
DDT1=(INCT1+REDT1-2*LF)/DA**2;
           X1=DDT1;
DDD=(INCD+REDD-2*LF)/DA**2;
                   Y2=DDD;
PDT1D=(INCT1D+REDT1D-2*LF-DA**2*(DDT1+DDD))/(2*DA**2);  X2=
   PDT1D;
/* DETERMINANT OF FISHER INFORMATION MATRIX*/
DET=X1*Y2-X2**2;
JP4=SQRT(DET);
FUND=T1*((JP4*PROBNORM(2*T1-D)**N112*PROBNORM(-2*T1-D)**N212
   *(PROBNORM(2*T1+D)-PROBNORM(2*T1-D))**N012))/NC;
FUN2=DL**2*FUND;
MT1+FUN2;
END;END;
PROC PRINT DATA=DD1;VAR DL NC MT1;RUN;
```

B. Program for Posterior Mean of θ_1 with UP for $m = 5$

```
DATA DD;
INPUT N012 N112 N212 N013 N113 N313 N023 N223 N323 N014
N114 N414 N015 N115 N515 N024 N224 N424 N025 N225 N525
N034 N334 N434 N035 N335 N535 N045 N445 N545 DA;
CARDS;
3 8 19 1 9 20 2 13 15 2 20 8 1 11 18 1 16 13 1 21
8 4 14 12 3 24 3 3 18 9 0.05;
PROC PRINT DATA=DD;  RUN;
DATA CC; SET DD;
DO T1=-4 TO 4-DA BY DA;
DO T2=-4 TO 4-DA BY DA;
DO T3=-4 TO 4-DA BY DA;
DO T4=-4 TO 4-DA BY DA;
DO D=0 TO 4-DA BY DA;
T5=-T1-T2-T3-T4;
```

```
Y112=PROBNORM(T1-T2-D); Y212=PROBNORM(T2-T1-D);
Y012=PROBNORM(T1-T2+D)-PROBNORM(T1-T2-D);
Y113=PROBNORM(T1-T3-D); Y313=PROBNORM(T3-T1-D);
Y013=PROBNORM(T1-T3+D)-PROBNORM(T1-T3-D);
Y114=PROBNORM(T1-T4-D); Y414=PROBNORM(T4-T1-D);
Y014=PROBNORM(T1-T4+D)-PROBNORM(T1-T4-D);
Y115=PROBNORM(T1-T5-D); Y515=PROBNORM(T5-T1-D);
Y015=PROBNORM(T1-T5+D)-PROBNORM(T1-T5-D);
Y223=PROBNORM(T2-T3-D); Y323=PROBNORM(T3-T2-D);
Y023=PROBNORM(T2-T3+D)-PROBNORM(T2-T3-D);
Y224=PROBNORM(T2-T4-D); Y424=PROBNORM(T4-T2-D);
Y024=PROBNORM(T2-T4+D)-PROBNORM(T2-T4-D);
Y225=PROBNORM(T2-T5-D); Y525=PROBNORM(T5-T2-D);
Y025=PROBNORM(T2-T5+D)-PROBNORM(T2-T5-D);
Y334=PROBNORM(T3-T4-D); Y434=PROBNORM(T4-T3-D);
Y034=PROBNORM(T3-T4+D)-PROBNORM(T3-T4-D);
Y335=PROBNORM(T3-T5-D); Y535=PROBNORM(T5-T3-D);
Y035=PROBNORM(T3-T5+D)-PROBNORM(T3-T5-D);
Y445=PROBNORM(T4-T5-D); Y545=PROBNORM(T5-T4-D);
Y045=PROBNORM(T4-T5+D)-PROBNORM(T4-T5-D);
PT=(Y112)**N112*(Y212)**N212*(Y012)**N012*
(Y113)**N113*(Y313)**N313*(Y013)**N013*
(Y114)**N114*(Y414)**N414*(Y014)**N014*
(Y115)**N115*(Y515)**N515*(Y015)**N015*
(Y223)**N223*(Y323)**N323*(Y023)**N023*
(Y224)**N224*(Y424)**N424*(Y024)**N024*
(Y225)**N225*(Y525)**N525*(Y025)**N025*
(Y334)**N334*(Y434)**N434*(Y034)**N034*
(Y335)**N335*(Y535)**N535*(Y035)**N035*
(Y445)**N445*(Y545)**N545*(Y045)**N045;
ppt=DA**5*PT; NC+PPT;
*OUTPUT; END; END; END; END; END;
PROC PRINT DATA=CC; VAR NC  DA;  RUN;
DATA  DD2;  SET CC; SET DD;
DO T1=-4 TO 4-DA BY DA;
DO T2=-4 TO 4-DA BY DA;
DO T3=-4 TO 4-DA BY DA;
DO T4=-4 TO 4DA BY DA;
DO D=0 TO 4-DA BY DA;
T5=-T1-T2-T3-T4;
Y112=PROBNORM(T1-T2-D); Y212=PROBNORM(T2-T1-D);
Y012=PROBNORM(T1-T2+D)-PROBNORM(T1-T2-D);
Y113=PROBNORM(T1-T3-D); Y313=PROBNORM(T3-T1-D);
Y013=PROBNORM(T1-T3+D)-PROBNORM(T1-T3-D);
Y114=PROBNORM(T1-T4-D); Y414=PROBNORM(T4-T1-D);
Y014=PROBNORM(T1-T4+D)-PROBNORM(T1-T4-D);
Y115=PROBNORM(T1-T5-D); Y515=PROBNORM(T5-T1-D);
Y015=PROBNORM(T1-T5+D)-PROBNORM(T1-T5-D);
Y223=PROBNORM(T2-T3-D); Y323=PROBNORM(T3-T2-D);
Y023=PROBNORM(T2-T3+D)-PROBNORM(T2-T3-D);
Y224=PROBNORM(T2-T4-D); Y424=PROBNORM(T4-T2-D);
```

```
Y024=PROBNORM(T2-T4+D)-PROBNORM(T2-T4-D);
Y225=PROBNORM(T2-T5-D); Y525=PROBNORM(T5-T2-D);
Y025=PROBNORM(T2-T5+D)-PROBNORM(T2-T5-D);
Y334=PROBNORM(T3-T4-D); Y434=PROBNORM(T4-T3-D);
Y034=PROBNORM(T3-T4+D)-PROBNORM(T3-T4-D);
Y335=PROBNORM(T3-T5-D); Y535=PROBNORM(T5-T3-D);
Y035=PROBNORM(T3-T5+D)-PROBNORM(T3-T5-D);
Y445=PROBNORM(T4-T5-D); Y545=PROBNORM(T5-T4-D);
Y045=PROBNORM(T4-T5+D)-PROBNORM(T4-T5-D);
JPM=T1*(Y112)**N112*(Y212)**N212*(Y012)**N012*
(Y113)**N113*(Y313)**N313*(Y013)**N013*
(Y114)**N114*(Y414)**N414*(Y014)**N014*
(Y115)**N115*(Y515)**N515*(Y015)**N015*
(Y223)**N223*(Y323)**N323*(Y023)**N023*
(Y224)**N224*(Y424)**N424*(Y024)**N024*
(Y225)**N225*(Y525)**N525*(Y025)**N025*
(Y334)**N334*(Y434)**N434*(Y034)**N034*
(Y335)**N335*(Y535)**N535*(Y035)**N035*
(Y445)**N445*(Y545)**N545*(Y045)**N045/NC;
TM4=DA**5*JPM;
MT1+TM4;
END;END;END;END;END;
PROC PRINT DATA=DD2; VAR DA NC MT1;RUN;
```

C. Program for Predictive Probability $P_(1.12)$ with UP for $m = 5$

```
DATA DD;
INPUT N012 N112 N212 N013 N113 N313 N023 N223 N323 N014 N114
N414 N015 N115 N515 N024 N224 N424 N025 N225 N525 N034 N334
N434 N035 N335 N535 N045 N445 N545 DA;
CARDS;
3 8 19 1 9 20 2 13 15 2 20 8 1 11 18 1 16 13 1 21
8 4 14 12 3 24 3 3 18 9 0.05
;
PROC PRINT DATA=DD;  RUN;
DATA CC; SET DD;
DO T1=-4 TO 4-DA BY DA;
DO T2=-4 TO 4-DA BY DA;
DO T3=-4 TO 4-DA BY DA;
DO T4=-4 TO 4-DA BY DA;
DO D=0 TO 4-DA BY DA;
T5=-T1-T2-T3-T4;
Y112=PROBNORM(T1-T2-D); Y212=PROBNORM(T2-T1-D);
Y012=PROBNORM(T1-T2+D)-PROBNORM(T1-T2-D);
Y113=PROBNORM(T1-T3-D); Y313=PROBNORM(T3-T1-D);
Y013=PROBNORM(T1-T3+D)-PROBNORM(T1-T3-D);
Y114=PROBNORM(T1-T4-D); Y414=PROBNORM(T4-T1-D);
Y014=PROBNORM(T1-T4+D)-PROBNORM(T1-T4-D);
Y115=PROBNORM(T1-T5-D); Y515=PROBNORM(T5-T1-D);
Y015=PROBNORM(T1-T5+D)-PROBNORM(T1-T5-D);
```

```
Y223=PROBNORM(T2-T3-D); Y323=PROBNORM(T3-T2-D);
Y023=PROBNORM(T2-T3+D)-PROBNORM(T2-T3-D);
Y224=PROBNORM(T2-T4-D); Y424=PROBNORM(T4-T2-D);
Y024=PROBNORM(T2-T4+D)-PROBNORM(T2-T4-D);
Y225=PROBNORM(T2-T5-D); Y525=PROBNORM(T5-T2-D);
Y025=PROBNORM(T2-T5+D)-PROBNORM(T2-T5-D);
Y334=PROBNORM(T3-T4-D); Y434=PROBNORM(T4-T3-D);
 Y034=PROBNORM(T3-T4+D)-PROBNORM(T3-T4-D);
Y335=PROBNORM(T3-T5-D); Y535=PROBNORM(T5-T3-D);
Y035=PROBNORM(T3-T5+D)-PROBNORM(T3-T5-D);
Y445=PROBNORM(T4-T5-D); Y545=PROBNORM(T5-T4-D);
Y045=PROBNORM(T4-T5+D)-PROBNORM(T4-T5-D);
PT=(Y112)**N112*(Y212)**N212*(Y012)**N012*(Y113)**N113*
(Y313)**N313*(Y013)**N013*(Y114)**N114*(Y414)**N414*
(Y014)**N014*(Y115)**N115*(Y515)**N515*(Y015)**N015*
(Y223)**N223*(Y323)**N323*(Y023)**N023*(Y224)**N224*
(Y424)**N424*(Y024)**N024*(Y225)**N225*(Y525)**N525*
(Y025)**N025*(Y334)**N334*(Y434)**N434*(Y034)**N034*
(Y335)**N335*(Y535)**N535*(Y035)**N035*(Y445)**N445*
(Y545)**N545*(Y045)**N045;
ppt=DA**5*PT; NC+PPT;
*OUTPUT; END; END; END; END; END;
PROC PRINT DATA=CC; VAR NC  DA;  RUN;
DATA  DD2;  SET CC; SET DD;
DO T1=-4 TO 4-DA BY DA;
DO T2=-4 TO 4-DA BY DA;
DO T3=-4 TO 4-DA BY DA;
DO T4=-4 TO 4-DA BY DA;
DO D=0 TO 4-DA BY DA;
T5=-T1-T2-T3-T4;
Y112=PROBNORM(T1-T2-D); Y212=PROBNORM(T2-T1-D);
Y012=PROBNORM(T1-T2+D)-PROBNORM(T1-T2-D);
Y113=PROBNORM(T1-T3-D); Y313=PROBNORM(T3-T1-D);
Y013=PROBNORM(T1-T3+D)-PROBNORM(T1-T3-D);
Y114=PROBNORM(T1-T4-D); Y414=PROBNORM(T4-T1-D);
Y014=PROBNORM(T1-T4+D)-PROBNORM(T1-T4-D);
Y115=PROBNORM(T1-T5-D); Y515=PROBNORM(T5-T1-D);
Y015=PROBNORM(T1-T5+D)-PROBNORM(T1-T5-D);
Y223=PROBNORM(T2-T3-D); Y323=PROBNORM(T3-T2-D);
Y023=PROBNORM(T2-T3+D)-PROBNORM(T2-T3-D);
Y224=PROBNORM(T2-T4-D); Y424=PROBNORM(T4-T2-D);
Y024=PROBNORM(T2-T4+D)-PROBNORM(T2-T4-D);
Y225=PROBNORM(T2-T5-D); Y525=PROBNORM(T5-T2-D);
Y025=PROBNORM(T2-T5+D)-PROBNORM(T2-T5-D);
Y334=PROBNORM(T3-T4-D); Y434=PROBNORM(T4-T3-D);
Y034=PROBNORM(T3-T4+D)-PROBNORM(T3-T4-D);
Y335=PROBNORM(T3-T5-D); Y535=PROBNORM(T5-T3-D);
Y035=PROBNORM(T3-T5+D)-PROBNORM(T3-T5-D);
Y445=PROBNORM(T4-T5-D); Y545=PROBNORM(T5-T4-D);
Y045=PROBNORM(T4-T5+D)-PROBNORM(T4-T5-D);
JPM=Y112*((Y112)**N112*(Y212)**N212*(Y012)**N012*
```

```
(Y113)**N113*(Y313)**N313*(Y013)**N013*
(Y114)**N114*(Y414)**N414*(Y014)**N014*
(Y115)**N115*(Y515)**N515*(Y015)**N015*
(Y223)**N223*(Y323)**N323*(Y023)**N023*
(Y224)**N224*(Y424)**N424*(Y024)**N024*
(Y225)**N225*(Y525)**N525*(Y025)**N025*
(Y334)**N334*(Y434)**N434*(Y034)**N034*
(Y335)**N335*(Y535)**N535*(Y035)**N035*
(Y445)**N445*(Y545)**N545*(Y045)**N045/NC);
TM4=DA**5*JPM;
P112+TM4;
END;END;END;END;END;
PROC PRINT DATA=DD2; VAR DA NC P112;RUN;
```

D. Program for Posterior Probability P_12 of H_12 with UP for $m = 5$

```
DATA DD;
INPUT N012 N112 N212 N013 N113 N313 N023 N223 N323
N014 N114 N414 N015 N115 N515 N024 N224 N424 N025
N225 N525 N034 N334 N434 N035 N335 N535 N045 N445 N545 DA;
CARDS;
3 8 19 1 9 20 2 13 15 2 20 8 1 11 18 1 16 13 1
21 8 4 14 12 3 24 3 3 18 9 0.05;
PROC PRINT DATA=DD;  RUN;
DATA CC; SET DD;
DO T1=-4 TO 4-DA BY DA;
DO T2=-4 TO 4DA BY DA;
DO T3=-4 TO 4-DA BY DA;
DO T4=-4 TO 4-DA BY DA;
DO D=0 TO 4-DA BY DA;
T5=-T1-T2-T3-T4;
Y112=PROBNORM(T1-T2-D); Y212=PROBNORM(T2-T1-D);
Y012=PROBNORM(T1-T2+D)-PROBNORM(T1-T2-D);
Y113=PROBNORM(T1-T3-D); Y313=PROBNORM(T3-T1-D);
Y013=PROBNORM(T1-T3+D)-PROBNORM(T1-T3-D);
Y114=PROBNORM(T1-T4-D); Y414=PROBNORM(T4-T1-D);
Y014=PROBNORM(T1-T4+D)-PROBNORM(T1-T4-D);
Y115=PROBNORM(T1-T5-D); Y515=PROBNORM(T5-T1-D);
Y015=PROBNORM(T1-T5+D)-PROBNORM(T1-T5-D);
Y223=PROBNORM(T2-T3-D); Y323=PROBNORM(T3-T2-D);
Y023=PROBNORM(T2-T3+D)-PROBNORM(T2-T3-D);
Y224=PROBNORM(T2-T4-D); Y424=PROBNORM(T4-T2-D);
Y024=PROBNORM(T2-T4+D)-PROBNORM(T2-T4-D);
Y225=PROBNORM(T2-T5-D); Y525=PROBNORM(T5-T2-D);
Y025=PROBNORM(T2-T5+D)-PROBNORM(T2-T5-D);
Y334=PROBNORM(T3-T4-D); Y434=PROBNORM(T4-T3-D);
Y034=PROBNORM(T3-T4+D)-PROBNORM(T3-T4-D);
Y335=PROBNORM(T3-T5-D); Y535=PROBNORM(T5-T3-D);
Y035=PROBNORM(T3-T5+D)-PROBNORM(T3-T5-D);
Y445=PROBNORM(T4-T5-D); Y545=PROBNORM(T5-T4-D);
```

```
Y045=PROBNORM(T4-T5+D)-PROBNORM(T4-T5-D);
PT=(Y112)**N112*(Y212)**N212*(Y012)**N012*
(Y113)**N113*(Y313)**N313*(Y013)**N013*
(Y114)**N114*(Y414)**N414*(Y014)**N014*
(Y115)**N115*(Y515)**N515*(Y015)**N015*
(Y223)**N223*(Y323)**N323*(Y023)**N023*
(Y224)**N224*(Y424)**N424*(Y024)**N024*
(Y225)**N225*(Y525)**N525*(Y025)**N025*
(Y334)**N334*(Y434)**N434*(Y034)**N034*
(Y335)**N335*(Y535)**N535*(Y035)**N035*
(Y445)**N445*(Y545)**N545*(Y045)**N045;
ppt=DA**5*PT; NC+PPT;
*OUTPUT; END; END; END; END; END;
PROC PRINT DATA=CC; VAR NC  DA;  RUN;
DATA  DD2;  SET CC; SET DD;
DO D=0 TO 4-DA BY DA;
DO T3=-4 TO 4-DA BY DA;
DO T4=-4 TO 4-DA BY DA;
DO ZZ=-4 TO 4-DA BY DA;
DO FF=0 TO 4-DA BY DA;
T1=ZZ; T2=ZZ-FF;
T5=-T1-T2-T3-T4;
Y112=PROBNORM(T1-T2-D); Y212=PROBNORM(T2-T1-D);
Y012=PROBNORM(T1-T2+D)-PROBNORM(T1-T2-D);
Y113=PROBNORM(T1-T3-D); Y313=PROBNORM(T3-T1-D);
Y013=PROBNORM(T1-T3+D)-PROBNORM(T1-T3-D);
Y114=PROBNORM(T1-T4-D); Y414=PROBNORM(T4-T1-D);
Y014=PROBNORM(T1-T4+D)-PROBNORM(T1-T4-D);
Y115=PROBNORM(T1-T5-D); Y515=PROBNORM(T5-T1-D);
Y015=PROBNORM(T1-T5+D)-PROBNORM(T1-T5-D);
Y223=PROBNORM(T2-T3-D); Y323=PROBNORM(T3-T2-D);
Y023=PROBNORM(T2-T3+D)-PROBNORM(T2-T3-D);
Y224=PROBNORM(T2-T4-D); Y424=PROBNORM(T4-T2-D);
Y024=PROBNORM(T2-T4+D)-PROBNORM(T2-T4-D);
Y225=PROBNORM(T2-T5-D); Y525=PROBNORM(T5-T2-D);
Y025=PROBNORM(T2-T5+D)-PROBNORM(T2-T5-D);
Y334=PROBNORM(T3-T4-D); Y434=PROBNORM(T4-T3-D);
Y034=PROBNORM(T3-T4+D)-PROBNORM(T3-T4-D);
Y335=PROBNORM(T3-T5-D); Y535=PROBNORM(T5-T3-D);
Y035=PROBNORM(T3-T5+D)-PROBNORM(T3-T5-D);
Y445=PROBNORM(T4-T5-D); Y545=PROBNORM(T5-T4-D);
Y045=PROBNORM(T4-T5+D)-PROBNORM(T4-T5-D);
JPM=(Y112)**N112*(Y212)**N212*(Y012)**N012*
(Y113)**N113*(Y313)**N313*(Y013)**N013*
(Y114)**N114*(Y414)**N414*(Y014)**N014*
(Y115)**N115*(Y515)**N515*(Y015)**N015*
(Y223)**N223*(Y323)**N323*(Y023)**N023*
(Y224)**N224*(Y424)**N424*(Y024)**N024*
(Y225)**N225*(Y525)**N525*(Y025)**N025*
(Y334)**N334*(Y434)**N434*(Y034)**N034*
(Y335)**N335*(Y535)**N535*(Y035)**N035*
```

```
(Y445)**N445*(Y545)**N545*(Y045)**N045/NC;
TM4=DA**5*JPM;
M5P12+TM4;
END;END;END;END;END;
PROC PRINT DATA=DD2; VAR DA NC M5P12;RUN;
```

E. Program for Graph of Marginal Posterior Mean of θ_1 with UP for$m = 5$

```
DATA DD;
INPUT N012 N112 N212 N013 N113 N313 N023 N223 N323 N014
N114 N414 N015 N115 N515 N024 N224 N424 N025 N225 N525
N034 N334 N434 N035 N335 N535 N045 N445 N545 DA;
CARDS;
3 8 19 1 9 20 2 13 15 2 20 8 1 11 18 1 16 13 1 21
8 4 14 12 3 24 3 3 18 9 0.05;
PROC PRINT DATA=DD;  RUN;
DATA CC; SET DD;
DO T1=-4 TO 4-DA BY DA;
DO T2=-4 TO 4-DA BY DA;
DO T3=-4 TO 4-DA BY DA;
DO T4=-4 TO 4-DA BY DA;
DO D=0 TO 4-DA BY DA;
T5=-T1-T2-T3-T4;
Y112=PROBNORM(T1-T2-D); Y212=PROBNORM(T2-T1-D);
Y012=PROBNORM(T1-T2+D)-PROBNORM(T1-T2-D);
Y113=PROBNORM(T1-T3-D); Y313=PROBNORM(T3-T1-D);
Y013=PROBNORM(T1-T3+D)-PROBNORM(T1-T3-D);
Y114=PROBNORM(T1-T4-D); Y414=PROBNORM(T4-T1-D);
Y014=PROBNORM(T1-T4+D)-PROBNORM(T1-T4-D);
Y115=PROBNORM(T1-T5-D); Y515=PROBNORM(T5-T1-D);
Y015=PROBNORM(T1-T5+D)-PROBNORM(T1-T5-D);
Y223=PROBNORM(T2-T3-D); Y323=PROBNORM(T3-T2-D);
Y023=PROBNORM(T2-T3+D)-PROBNORM(T2-T3-D);
Y224=PROBNORM(T2-T4-D); Y424=PROBNORM(T4-T2-D);
Y024=PROBNORM(T2-T4+D)-PROBNORM(T2-T4-D);
Y225=PROBNORM(T2-T5-D); Y525=PROBNORM(T5-T2-D);
Y025=PROBNORM(T2-T5+D)-PROBNORM(T2-T5-D);
Y334=PROBNORM(T3-T4-D); Y434=PROBNORM(T4-T3-D);
Y034=PROBNORM(T3-T4+D)-PROBNORM(T3-T4-D);
Y335=PROBNORM(T3-T5-D); Y535=PROBNORM(T5-T3-D);
Y035=PROBNORM(T3-T5+D)-PROBNORM(T3-T5-D);
Y445=PROBNORM(T4-T5-D); Y545=PROBNORM(T5-T4-D);
Y045=PROBNORM(T4-T5+D)-PROBNORM(T4-T5-D);
PT=(Y112)**N112*(Y212)**N212*(Y012)**N012*
(Y113)**N113*(Y313)**N313*(Y013)**N013*
(Y114)**N114*(Y414)**N414*(Y014)**N014*
(Y115)**N115*(Y515)**N515*(Y015)**N015*
(Y223)**N223*(Y323)**N323*(Y023)**N023*
(Y224)**N224*(Y424)**N424*(Y024)**N024*
```

```
(Y225)**N225*(Y525)**N525*(Y025)**N025*
(Y334)**N334*(Y434)**N434*(Y034)**N034*
(Y335)**N335*(Y535)**N535*(Y035)**N035*
(Y445)**N445*(Y545)**N545*(Y045)**N045;
ppt=DA**5*PT; NC+PPT;
*OUTPUT; END; END; END; END; END;
PROC PRINT DATA=CC; VAR NC  DA;  RUN;
DATA  DD2;  SET CC; SET DD; DA1=0.01;
DO T1=-4 TO 4-DA1 BY DA1; MT1=0;
DO T2=-4 TO 4-DA BY DA;
DO T3=-4 TO 4-DA BY DA;
DO T4=-4 TO 4-DA BY DA;
DO D=0 TO 4-DA BY DA;
T5=-T1-T2-T3-T4;
Y112=PROBNORM(T1-T2-D); Y212=PROBNORM(T2-T1-D);
Y012=PROBNORM(T1-T2+D)-PROBNORM(T1-T2-D);
Y113=PROBNORM(T1-T3-D); Y313=PROBNORM(T3-T1-D);
Y013=PROBNORM(T1-T3+D)-PROBNORM(T1-T3-D);
Y114=PROBNORM(T1-T4-D); Y414=PROBNORM(T4-T1-D);
Y014=PROBNORM(T1-T4+D)-PROBNORM(T1-T4-D);
Y115=PROBNORM(T1-T5-D); Y515=PROBNORM(T5-T1-D);
Y015=PROBNORM(T1-T5+D)-PROBNORM(T1-T5-D);
Y223=PROBNORM(T2-T3-D); Y323=PROBNORM(T3-T2-D);
Y023=PROBNORM(T2-T3+D)-PROBNORM(T2-T3-D);
Y224=PROBNORM(T2-T4-D); Y424=PROBNORM(T4-T2-D);
Y024=PROBNORM(T2-T4+D)-PROBNORM(T2-T4-D);
Y225=PROBNORM(T2-T5-D); Y525=PROBNORM(T5-T2-D);
Y025=PROBNORM(T2-T5+D)-PROBNORM(T2-T5-D);
Y334=PROBNORM(T3-T4-D); Y434=PROBNORM(T4-T3-D);
Y034=PROBNORM(T3-T4+D)-PROBNORM(T3-T4-D);
Y335=PROBNORM(T3-T5-D); Y535=PROBNORM(T5-T3-D);
Y035=PROBNORM(T3-T5+D)-PROBNORM(T3-T5-D);
Y445=PROBNORM(T4-T5-D); Y545=PROBNORM(T5-T4-D);
Y045=PROBNORM(T4-T5+D)-PROBNORM(T4-T5-D);
JPM=T1*(Y112)**N112*(Y212)**N212*(Y012)**N012*
(Y113)**N113*(Y313)**N313*(Y013)**N013*
(Y114)**N114*(Y414)**N414*(Y014)**N014*
(Y115)**N115*(Y515)**N515*(Y015)**N015*
(Y223)**N223*(Y323)**N323*(Y023)**N023*
(Y224)**N224*(Y424)**N424*(Y024)**N024*
(Y225)**N225*(Y525)**N525*(Y025)**N025*
(Y334)**N334*(Y434)**N434*(Y034)**N034*
(Y335)**N335*(Y535)**N535*(Y035)**N035*
(Y445)**N445*(Y545)**N545*(Y045)**N045/NC;
TM4=DA**5*JPM;
MT1+TM4;
END;END;END;END;OUTPUT;END;
PROC PRINT DATA=DD2; VAR DA NC MT1;RUN;
axis1 length=2 in;
axis2 length=4 in;
symbol1 interpol=join;
```

```
proc gplot DATA=DD2;
plot MT1*T1=1/vaxis=axis1 haxis=axis2;RUN;
```

References

Altaf, S., Aslam, M., & Aslam, M. (2013). Bayesian analysis of the amended Davidson model for paired comparison using non-informative and informative priors. *Statistics*, *47*(5), 1090–1103. https://doi.org/10.1080/02331888.2012.664773

Aslam, M., & Cheema, A. N. (2020). Bayesian analysis for 3-component mixture of exponentiated Weibull distribution assuming non-informative priors. *Journal of Statistical Computation and Simulation*, *90*(4), 586–605. https://doi.org/10.1080/00949655.2019.1692840

Aslam, M., & Kifayat, T. (2018). Bayesian analysis of the Rayleigh paired comparison model under loss functions using an informative prior. *Scientia Iranica*, *25*(2), 983–990. https://doi.org/10.24200/sci.2017.4438

Aslam, M., & Shah, S. H. (2015). Lindley-Shannon information for comparison of priors under paired comparisons model. *Pakistan Journal of Statistics and Operation Research*, *11*(3), 317–329. https://doi.org/10.18187/pjsor.v11i3.779

Aslam, M. (1996). *Bayesian analysis for paired comparisons data* [Ph.D. Dissertation]. University of Wales, Aberystwyth.

Awan, K. U., & Aslam, M. (2020). Bayesian analysis of the Weibull paired comparison model using numerical approximation. *Journal of Mathematics*, *2020*(1), 1–6. https://doi.org/10.1155/2020/6628379

Beaudoin, D., & Swartz, T. (2018). A computationally intensive ranking system for paired comparison data. *Operations Research Perspectives*, *5*, 105–112. https://doi.org/10.1016/j.orp.2018.03.002

Csató, L. (2015). A graph interpretation of the least squares ranking method. *Social Choice and Welfare*, *44*(1), 51–69. https://doi.org/10.48550/arXiv.1508.06778

Glenn, W. A., & David, H. A. (1960). Ties in paired comparison experiments using a modified Thurstone–Mosteller model. *Biometrics*, *16*(1), 86–109. https://doi.org/10.2307/2527957

Liu, K. H., & Shih, Y. S. (2016). Score-scale decision tree for paired comparison data. *Statistica Sinica*, *26*, 429–444. https://doi.org/10.5705/ss.2014.164

Mosteller, F. (1951). Remarks on the method of paired comparisons: II. The effect of an aberrant standard deviation when equal standard deviations and equal correlations are assumed. *Psychometrika*, *16*, 203–206. https://doi.org/10.1007/BF02289115

Sindhu, N. T., Hussain, Z., & Aslam, M. (2019). Parameter and reliability estimation of inverted Maxwell mixture model. *Journal of Statistics & Management Systems*, *22*(3), 459–493. https://doi.org/10.1080/09720510.2018.1552412

Thurstone, L. L. (1927). Psychophysical analysis. *American Journal of Psychology*, *38*, 369–389.

Tian, X., Shi, J., Shen, X., & Song, K. (2024). A spectral approach for the dynamic Bradley–Terry model. *Stat*, *13*(3), e722. https://doi.org/10.1002/sta4.722

Tutz, G., & Schauberger, G. (2015). Extended ordered paired comparison models with application to football data from German Bundesliga. *AStA Advances in Statistical Analysis*, *99*(2), 209–227. https://doi.org/10.1007/s10182-014-0237-1

Varin, C., & Firth, D. (2024). Ridge regression for paired comparisons: A tractable new approach, with application to Premier League football. *arXiv preprint arXiv:2406.09597*. https://doi.org/10.48550/arXiv.2406.09597